Quantum Error Correction

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1. Set up

Input state $\rho \xrightarrow[\text{error}]{\text{noise,}} \hat{\mathcal{E}}(\rho) \xrightarrow[\text{Correction}]{\text{Error}} \hat{R}\circ\hat{\mathcal{E}}(\rho) = \rho$

$\hat{\mathcal{E}}$: Superoperator $\hat{R}$: Recovery

For any arbitrary $\rho \in \mathcal{H}^{\otimes n}$.

$\hat{R}\circ\hat{\mathcal{E}} = I$ is impossible.

– Quantum codes C: subspace of $\mathcal{H}^{\otimes n}$, s.t.
$\hat{R}\circ\hat{\mathcal{E}}(\rho) = \rho$, $\forall \rho \in C$.

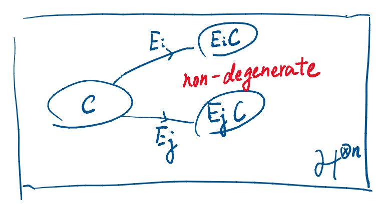
Theorem 1.
Let C be a quantum code; P be a projector onto C.

The errors $\{E_i\}$ are correctable ($\exists \hat{R}$ s.t. $P\hat{R}\circ\hat{\mathcal{E}}P = P\hat{I}P$)
if and only if $P E_i^\dagger E_j P = \alpha_{ij} P$

Def. $\hat{\mathcal{E}}(\rho) = \sum_i E_i \rho E_i^\dagger$

$\alpha_{ij} \propto \delta_{ij}$. non-degenerate code
$\alpha_{ij} = c$. completely degenerate

Proof. (Intuition).



Theorem 2.
If errors $\{E_i\}$ are correctable, any superposition
$F_j = \sum_i \beta_{ji} E_i$ are also correctable.

Proof. $P F_j^\dagger F_{j'} P = \sum_{i,i'} \beta_{ij}\beta_{j'i'} P E_i^\dagger E_{i'} P$

Remark:
For single-bit error,
consider only X. Y. Z. I.

Theorem 3. (Quantum Hamming Bound).

One needs at least 5 bits to encode 1 bit of information.
to correct arbitrary single-bit errors.

Proof. For n bits, # of possible independent single-bit errors. $= 3n+1$

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n: physical bits. 1-error, 0-error.

$C \xrightarrow{E_i} E_iC$

$\dim(C) = 2^k$ $\longrightarrow$ k: logical bits

$\Rightarrow (3n+1)\cdot \dim C \le \dim(\mathcal{H}^n)$ $\Rightarrow$ for $k=1$, $n\ge 5$.

of ◯s, 2^k, 2^n

$(5, 1, 1)$ code

(n, k, t) code — physical, logical, correctable errors.

(n, k, d), $d = 2t+1$ — Hamming distance between subspaces

2. Stabilizer codes

- Pauli groups $\langle \pm i, I, X_i, Y_j, Z_k \rangle$

- Stabilizer operator $\{g_i\} \in P$ with g_i commute with each other.

For n bits, n independent stabilizers $\Rightarrow$ stabilizer state

$k<n$ independent stabilizers $\Rightarrow$ fix a subspace C, $\dim C = 2^{n-k}$.

stabilizer code state.

Single-bit errors $E_\alpha = \{X_i, Y_j, Z_k \mid 1\le i,j,k \le n\}$.

code C. $g_i C = C$. $(1 \le i \le k)$.

with error. $g_i E_\alpha C = \pm E_\alpha g_i C = \pm E_\alpha C$.

Error Syndrone.

	g_1	g_2		$\cdots$	g_k
C	1	1		$\cdots$	1
$E_\alpha C$	1	-1		$\cdots$	1.
$E_\alpha' C$	1	-1	-1	$\cdots$	1

For different error, error syndrome must be different

$\Rightarrow R = E_\alpha^{-1} = E_\alpha$

Example. $(n=5, r=1, t=1)$

$$\begin{cases} g_1: X_1 Z_2 Z_3 X_4 I_5 \\ g_2: I_1 X_2 Z_3 Z_4 X_5 \\ g_3: X_1 I_2 X_3 Z_4 Z_5 \\ g_4: Z_1 X_2 I_3 X_4 Z_5 \end{cases}$$

	g_1	g_2	g_3	g_4
C	1	1	1	1
E_α	1	-1	1	1
	$\cdots$		$\cdots$	

$2^4 = 16 = 3\times 5+1$

$\longrightarrow g_i E_\alpha C$ to find $\underline{E_\alpha}$.

but $[g_i, X_L] = 0$. $[g_i, Z_L] = 0$

$\Rightarrow$ original state not changed.

$\{ Z_L: Z\,Z\,Z\,Z\,Z$ acting on logic-qubit $\Rightarrow C = \{c_0|0\rangle_L + c_1|1\rangle_L\}$

$d4'$ Z_1 X_2 I_3 X_4 Z_5

$$\begin{cases} Z_L: Z\ Z\ Z\ Z\ Z \\ X_L: X\ X\ X\ X\ X \end{cases}$$

acting on logic-qubit $\Rightarrow$ $C = \{C_0|0\rangle_L + C_1|1\rangle_L\}$

$X_L|0\rangle_L = |1\rangle_L$. $X_L|1\rangle = |0\rangle_L$

$Z_L|0\rangle_L = |0\rangle_L$. $Z_L|1\rangle = -|1\rangle_L$

Encoded X_L, Z_L

H_L, S_L, $CNOT_L$, T_L

Fault-tolerant Q.C.

1. Basic Idea.

$\begin{cases} \text{state prep.} \\ \text{universal gate} \\ \text{measurement} \end{cases}$ have error p.

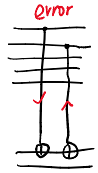
Independent noise model. with n element $\begin{cases} \text{one-bit error } \binom{n}{1} p = np \\ \text{two-bit error } \binom{n}{2} p^2 \end{cases}$

Code block.

block 1
(logic qubit)

block 2.
(logic ...).

error

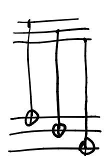


1 error → 2 error

amplify the error X.

block 1

block 2.



(fault-tolerant) implementation.

1 error

$\Rightarrow$ 1 error in each block

Suppose n faulty elements are done fault-tolerantly

If one element fails $\Rightarrow$ one error in each encoding block.

two element fails $\Rightarrow$ cannot correct

$\binom{n}{2} p^2$ remaining error given by $O(p^2)$.
$= cp^2$

one physical elem. $\xrightarrow{Q.E.C.}$ n physical elems

p error $\qquad\qquad$ cp^2 error.

We want $p < \frac{1}{c} \equiv P_{th}$ (threshold)

$$\boxed{G} \Longrightarrow \boxed{G_L}$$

Concatenation:

$$\underset{p}{\boxed{G}} \longrightarrow \underset{cp^2}{\boxed{G_L}} \longrightarrow \underset{c(cp^2)^2}{\boxed{G_L'}}$$

$\xrightarrow{\text{k-level encoding}}$ remaining error $= \frac{(cp)^{2^k}}{c}$ $(= P_f^{(k)})$

Total cost of physical bits : d^k.

Total error:

$$n \cdot P_f^{(k)} = n \cdot \frac{(cp)^{2^k}}{c} \leq \varepsilon. \quad \Rightarrow 2^k = \log_2\left(\frac{c\varepsilon}{n}\right) \Big/ \log_2(cp).$$

$$d^k = 2^{k \log_2 d}$$

$$\doteq \left(\frac{\log\left(\frac{n}{c\varepsilon}\right)}{\log\left(\frac{1}{cp}\right)} \right)^{\log d} = O\left(\mathrm{poly}\left(\frac{n}{\varepsilon}\right)\right)$$