Quantum Entanglement Theory

Thursday, March 26, 2020 10:16 AM

· Entanglement for pure states.

1. $|\psi_{AB}\rangle$ $\quad \rho_A = tr_B(|\psi\rangle_{AB}\langle\psi|)$

$$E(|\psi\rangle_{AB}) = S(\rho_A) = -tr(\rho_A \log \rho_A) = S(\rho_B)$$

2. Entanglement of formation and distillation.

LOCC operators $\begin{cases} \text{local operations.} \\ \text{classical communication.} \end{cases}$

LOCC does not increase entanglement

How to quantify entanglement ?

– Compare it to $|\Phi^+\rangle_{AB} = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$

Single - copy operation. $\quad |\psi\rangle_{AB} \xrightarrow[?]{LOCCs} |\Phi^+\rangle_{AB}$

Asymtotic operation $\quad |\psi\rangle_{AB}^{\otimes n} \underset{\longleftarrow}{\xrightarrow{LOCCs}} |\Phi^+\rangle_{AB}^{\otimes k}$

Entanglement of formation E_F

$$E_F \equiv \lim_{n\to\infty} \frac{k_{min}}{n}$$

(the least number of $|\Phi^+\rangle_{AB}$ required to prepare n compies of $|\psi\rangle_{AB}$ by LOCC).

$$|\psi\rangle_{AB}^{\otimes n} \xrightarrow{LOCC} |\Phi^+\rangle_{AB}^{\otimes k}$$

Entanglement of distillation

$$E_D(|\psi\rangle_{AB}) \equiv \lim_{n\to\infty} \frac{k_{max}}{n}$$

$$\Rightarrow \quad E_F(|\psi\rangle_{AB}) \geqslant E_D(|\psi\rangle_{AB}).$$

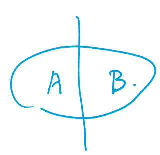
<u>Theorem</u> For any bipartite pure states $|\psi\rangle_{AB}$.

$$E_F(|\psi\rangle_{AB}) = E_D(|\psi\rangle_{AB}) = S(\rho_A)$$

Proof idea $\begin{cases} \text{Quantum teleportation} \\ \text{Quantum data compression.} \end{cases}$

3. Multi - partite entanglement.

–. Block entanglement : N-particle $\longrightarrow$ divide to 2 blocks then



$$E_{Block} = S(\rho_A)$$

— Reduced density ρ_{AB} (change N to 2).

localizable entanglement : Entanglement of ρ_{AB} after optimized local operation on other partites.

Entanglement for mixed states

1. Entanglement criterion

$$\underline{\rho_{AB} = \sum_i p_i \, \rho_{iA} \otimes \rho_{iB}} \quad \text{(separable state).}$$

Def. A mixed state is called entangled state iff it cannot be written into a separable form

Entangled ?

For pure state $S(\rho_A) \neq 0$

For mixed state $S(\rho_A)$ does not work.

Ex. $\rho_{AB} = p|\Phi^+\rangle\langle\Phi^+| + (1-p)|\Phi^-\rangle\langle\Phi^-|$

when $p = \frac{1}{2}$ separable

2. Theorem

Partial transpose criterion.

$\forall$ separable state ρ_{AB}, the partial transpose $\rho_{AB}^{T_A}$ is positive.

$$\rho_{AB}^{T_A} = \sum_i p_i \, \rho_{iA}^{T_A} \otimes \rho_{iB}$$
$$= \sum_i p_i \, \rho_{iA}^{*} \otimes \rho_{iB}$$
$$\geq 0$$

eigenvalues are the same, $\rho_{iA}^* \geq 0$

if ρ_{AB} is 2x2 or 2x3, it is sufficient condition as well.

Ex 1

$|\phi\rangle^+ = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$

$$\rho_{AB} = \frac{1}{2}\left[\,|00\rangle\langle 00| + |11\rangle\langle 11| + |00\rangle\langle 11| + |11\rangle\langle 00|\,\right]$$

$$\rho_{AB}^{T_B} = \frac{1}{2}\left[\,|00\rangle\langle 00| + |11\rangle\langle 11| + |01\rangle\langle 10| + |10\rangle\langle 01|\,\right]$$

$$= \frac{1}{2}\begin{pmatrix} 1 & & & \\ & 0 & 1 & \\ & 1 & 0 & \\ & & & 1 \end{pmatrix} < 0. \quad \Rightarrow \text{entangled}$$

Ex 2.

$$\rho_{AB} = p\,|\Phi^+\rangle_{AB}\langle\Phi^+| + \frac{1-p}{4} I_{AB}$$

$$= \frac{p}{2}\begin{bmatrix} 1 & & & \\ & 0 & 1 & \\ & 1 & 0 & \\ & & & 1 \end{bmatrix} + \frac{1-p}{4}\begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{bmatrix}$$

$$= \begin{pmatrix} \frac{1+p}{4} & & & \\ & \frac{1-p}{4} & \frac{p}{2} & \\ & \frac{p}{2} & \frac{1-p}{4} & \\ & & & \frac{1+p}{4} \end{pmatrix}$$

$\frac{1-p}{4} > \frac{p}{2} \Leftrightarrow$ positive

If $p > \frac{1}{3} \Leftrightarrow$ entangled

Remarks:

1. Positive partial transpose states (PPT) $(\rho_{AB}^{T_B} \geq 0) \Leftrightarrow$ separable state for 2x2, 2x3 systems.

   NPPT: $\rho_{AB}^{T_B} < 0 \Rightarrow$ entangled state

2. $\exists$ PPT but entangled for higher dimension $(E_F > 0, \; E_D = 0)$

   $\Rightarrow$ Bound entangled states

3. Entanglement Measurement.

— General criteria for measurements E:

1) $E(\rho_{AB}) = 0$ iff ρ_{AB} separable.

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1) $E(\rho_{AB}) = 0$ iff ρ_{AB} separable.

2) $E(U_A \otimes U_B\, \rho_{AB}\, U_A^\dagger \otimes U_B^\dagger) = E(\rho_{AB})$

3) $E(\rho_{AB})$ non-increasing for LOCC. (Entanglement monotones)

4) Reduce to Von-Neumann Entropy when ρ_{AB} is a pure state

1) Entanglement of formation and distillation

$E_F \geq E_D$. For pure state (bi-partite) $E_F = E_D$.

$E_F > E_D = 0$ for bound entangled state.

For 2x2 (2-qubit) systems : $\boxed{\text{Wootter's formula}}$ (to bound E_F in mixed state).

Def. Entanglement cost E_c.

For ρ_{AB} : $\exists$ ensemble decomposition : $\rho_{AB} = \sum_i P_i |\psi_i\rangle_{AB}\langle\psi_i|$

$$E_c(\rho_{AB}) = \underset{\substack{\uparrow \\ \text{over all ensemble decomposition.}}}{\min} \sum_i P_i \underbrace{E(|\psi_i\rangle_{AB})}_{S(\rho_A)}$$

$E_F \leq E_c$

To calculate E_c :

Def. $\tilde{\rho}_{AB} = (Y_A \otimes Y_B)\, \rho_{AB}^{*}\, (Y_A \otimes Y_B)$.

complex conjugate

$Y \equiv \sigma_y$

$$R = \sqrt{\sqrt{\rho_{AB}}\, \tilde{\rho}_{AB}\, \sqrt{\rho_{AB}}}$$

Remark:

Fidelity prepare (pure) $|\psi\rangle_{AB}$, real ρ_{AB}

$F \overset{def.}{=} {}_{AB}\langle\psi|\rho_{AB}|\psi\rangle_{AB}$ The difference between target and real

If target state is mixed : $\tilde{\rho}_{AB}$

The Fidelity $F = tr(R)^2$

Then, Find eigenvalues of R: $\lambda_1 \geq \lambda_2 \geq \lambda_3 \geq \lambda_4$. (Real)

problems.

E_c for Bell diagonal states.

$\rho_{AB} = P_1 |\Phi^+\rangle_{AB}\langle\Phi^+| + P_2 |\Phi^-\rangle_{AB}\langle\Phi^-|$
$+ P_3 |\Psi^+\rangle_{AB}\langle\Psi^+|_{AB} + P_4 |\Psi^-\rangle_{AB}\langle\Psi^-|$

Def. Concurrence.

$$C(\rho_{AB}) \overset{def.}{=} \max\{0, \lambda_1 - \lambda_2 - \lambda_3 - \lambda_4\}.$$

$$\Rightarrow \quad E_c(\rho_{AB}) = h\left(\frac{1+\sqrt{1-C^2(\rho_{AB})}}{2}\right), \quad \text{where } h(x) \overset{def.}{=} -x\log_2 x - (1-x)\log_2(1-x)$$

(Shannon Entropy)

2) Negativity

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$$N(\rho_{AB}) = \sum_i \frac{|\lambda_i| - \lambda_i}{2}, \quad \lambda_i \text{ eigenvalues of } \rho_{AB}^{T_B}$$

$$= \left| \text{sum of all negative eigenvalues of } \rho_{AB}^{T_B} \right|$$

Logrithmic : $LN(\rho_{AB}) = \log_2 \left[2N(\rho_{AB}) + 1 \right]$

Both N and LN are entanglement measurements

Upper Bound : $E_D(\rho_{AB}) \leq LN(\rho_{AB})$

In particular, PPT entangled states : $LN(\rho_{AB}) = N(\rho_{AB}) = 0$

$\Rightarrow E_D(\rho_{AB}) = 0$

Remark : 1) If $E_1(\rho_{1AB}) > E_1(\rho_{2AB})$

Then $E_2(\rho_{1AB}) \cancel{>} E_1(\rho_{2AB})$

Not neccessarily True !

(For mixed state)

Typical Entangled States

1. EPR, Bell states (2-qubit)

EPR / Bell : $|\Phi^{\pm}\rangle = \frac{1}{\sqrt{2}}(|00\rangle \pm |11\rangle)$

$|\Psi^{\pm}\rangle = \frac{1}{\sqrt{2}}(|10\rangle \pm |00\rangle)$

Locally equivalent

Bell diagonal $\rho_B = P_1 |\Phi^+\rangle\langle\Phi^+| + P_2 |\Phi^-\rangle\langle\Phi^-| + P_3 |\Psi^+\rangle\langle\Psi^-| + P_4 |\Psi^-\rangle\langle\Psi^-|$

when $P_2 = P_3 = P_4 \Rightarrow$ Werner state

2. GHZ, W multi-qubit.

$$|GHZ_3\rangle = \frac{1}{\sqrt{2}}(|000\rangle + |111\rangle)$$

$$|GHZ_N\rangle = \frac{1}{\sqrt{2}}(|0\rangle^{\otimes n} + |1\rangle^{\otimes n}).$$

(or Schrödinger Cat State)

$$|W\rangle_3 = \frac{1}{\sqrt{3}}(|001\rangle + |010\rangle + |100\rangle)$$

$$|W\rangle_n = \frac{1}{\sqrt{n}}(|00\cdots1\rangle + |00\cdots10\rangle + \cdots + |100\cdots0\rangle)$$

3. Cluster, Graph, Stabilizer states.

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Define entangled states as co-eigenstates of a set of $\underbrace{\text{commuting operators}}_{\text{Stabilizers}}$.

- How to find $\{A_i\}$ and make them commutable?

Pauli group: elements are tensor product of Pauli operators I_i, X_i, Y_i, Z_i, $\underset{=}{i}$ qubits

Properties:
1. any two elements of Pauli group either commute or anti-commute.
2. $[X_i Z_j, X_j Z_i] = 0$
3. $A_i^2 \in I$ $\Rightarrow$ A_i has eigenvalue ± 1.

Ex.

$A_1 = X_1 Y_2 Z_3 X_4$

$A_2 = Y_1 X_2 Y_3 Y_4$

$\rightarrow X_1 X_2 = X_1 \otimes X_2$ (by default)

$X_1 Y_1 = X_1 Y_1$

# of difference : 4 $\rightarrow$ even $\rightarrow$ commute

if odd $\rightarrow$ anti-commute.

Def. Cluster states : co-eigenstates of A_i with +1 eigenvalues where A_i is associate with a lattice.

Ex.

$X_1 Z_2 \quad X_2 Z_1 Z_3 \quad \cdots$

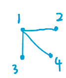
$\uparrow \qquad \uparrow$

$A_1 \quad A_2 \quad \cdots$

$$A_i \overset{def}{=} X_i \bigotimes_{\substack{j \in N(i) \\ \text{neighbours.}}} Z_j$$

Def. Graph states : associate with an arbitrary graph $G \equiv (\underset{\text{vertex}}{V}, \underset{\text{edge}}{E})$.

Ex.



$$A_i \overset{def}{=} X_i \bigotimes_{j \in N(i)} Z_j$$

Remark :
1) n qubits state uniquely determined by n stabilizer opertors.
2) Any stabilizer states of Pauli group can be written as a graph state by local unitary transformation

Ex. $X_1 X_2, \; Z_1 Z_2 \xRightarrow{Y_2} X_1 Z_2, \; X_2 Z_1$

State $\longleftarrow$ stabilizers $\rightleftharpoons$ graph states.

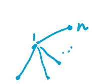
Ex. $X_1 Z_2 \quad X_2 Z_1$

$$\begin{cases} X_1 Z_2 |\psi\rangle = |\psi\rangle \\ X_2 Z_1 |\psi\rangle = |\psi\rangle \end{cases}$$

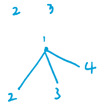
$$\Rightarrow |\psi\rangle = \frac{1}{\sqrt{2}} \left(|0\rangle_1 |+\rangle_2 + |1\rangle_1 |-\rangle_2 \right.$$

$$|\pm\rangle = \frac{1}{\sqrt{2}} (|0\rangle \pm |1\rangle)$$

$X_1 Z_2 \quad X_2 Z_1 Z_3 \quad X_3 Z_2 \xRightarrow{\text{Local unitary}} GHZ_3$



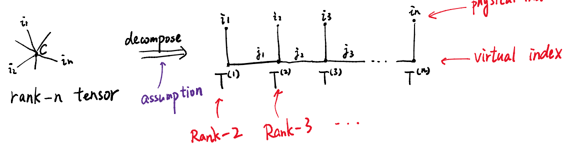
$\xRightarrow{\text{LU equivalent}} |0\rangle^{\otimes n} + |1\rangle^{\otimes n}$



$\xrightarrow{\text{not LU equiv.}}$

4. Tensor network state.

$$|\psi\rangle = \sum_{i_1 i_2 \cdots i_n} \underbrace{C_{i_1 i_2 \cdots i_n}}_{2^n \text{ coefficient}} |i_1 i_2 \cdots i_n\rangle.$$

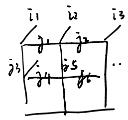


$$\Rightarrow C_{i_1 i_2 \cdots i_n} = \sum_{j_1 j_2 \cdots j_{n-1}} T^{(i_1)}_{j_1} T^{(i_2)}_{j_1 j_2} \cdots T^{(i_n)}_{j_{n-1}}$$ $\Rightarrow$ Matrix product state.

Contraction

↓ density matrix renormalization group approach (DMRG)

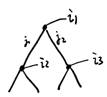
2D $\Rightarrow$



PEPS

→ NP-hard

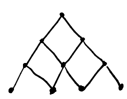
Tree tensor network



$$\sum_{j_1 \cdots j_n} T^{(i_1)}_{j_1 j_2} T^{(i_2)}_{j_1 j_3 j_4} \cdots$$

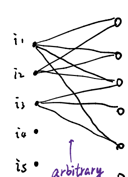
→ efficient contraction $\Leftrightarrow$ tree width small.

MERA (multi-scale entanglement renormalization ansatz) states



5. Neural Network State.

Boltzmann machine (restricted)



visible neurons (real qubits)

hidden neurons

weight function for each edge (i,j)

$$C_{i_1 i_2 \cdots i_n} = \sum_{j_1 j_2 \cdots j_n} e^{w(i_1,j_1) + w(i_2,j_2) + \cdots}$$

$w(i,j) = W_{11} ij + W_{01} i + W_{10} j + W_{00}$

(since $i, j \in \{0,1\}$)

W_{ij} : real values $\Rightarrow$ RBM.

complex $\Rightarrow$ quantum RBM

Quantum nonlocality and Bell's Inequality

1. Background: EPR paradox & hidden variable Theory

$|\psi\rangle_{AB}$: a co-eigenstate of $X_A X_B$, $Z_A Z_B$ with $(+1,+1)$ eigenvalue

Alice —— $|\psi\rangle_{AB}$ —— Bob.

measure X_A $\Rightarrow +1$ $\Rightarrow X_B : +1$

measure Z_B : -1 → Z_A : -1

$\Rightarrow \left\{ \begin{array}{l} X_A \\ Z_A \end{array} \right.$ known simultaneously?

measure Z_B

measure X_A $\Rightarrow +1$ $\Rightarrow X_B : +1$

$Z_A : -1$

$\Rightarrow \begin{cases} X_A \\ Z_A \end{cases}$ known simultaneously. ?

Quantum: objectivity

Observables do not have pre-determined values before measuring

Hidden varlable theory

Observables fundamental. $\Longleftrightarrow$ physical reality (pre-assigned value)

Local hidden variable theory: $\begin{cases} \text{Observables have pre-assigned values. satisfies classical prob. Theory} \\ \text{Einstein's locality (no superliminal interaction).} \end{cases}$

(LHV)

2. Bell's Inequality (CHSH)

Alice | Bob

Experiment facts: $\begin{cases} A_1 \ \pm 1 \\ A_2 \ \pm 1 \end{cases}$ $\left.\begin{matrix} B_1 \ \pm 1 \\ B_2 \ \pm 1 \end{matrix}\right\}$ facts.

↳ Random variables.

define: $W = (A_1+A_2)B_1 - (A_1-A_2)B_2$

Possible values: if $A_1+A_2 = \pm 2$, $A_1 - A_2 = 0$ / if $A_1+A_2 = 0$, $A_1 - A_2 = \pm 2$ $\Rightarrow W = \pm 2$ ← if observable <u>has</u> pre-determined value. (LHV)

so $|\langle W \rangle| \le 2$ ← A doesn't influence B (LHV)

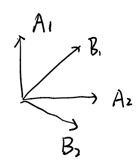
Bell-CHSH inequality

But for Q.M. $\exists$ state to violate Bell's Inequality

$\Rightarrow$ Example: $|\psi^-\rangle = \frac{\sqrt{2}}{2}(|01\rangle - |10\rangle)$

$$(\vec{\sigma}_A + \vec{\sigma}_B)|\psi^-\rangle_{AB} = 0.$$

$$\begin{aligned} &{}_{AB}\langle\psi^-|(\vec{n}\cdot\vec{\sigma}_A)(\vec{m}\cdot\vec{\sigma}_B)|\psi^-\rangle_{AB} \\ =\ &{}_{AB}\langle\psi^-|-(\vec{n}\cdot\vec{\sigma}_A)(\vec{m}\cdot\vec{\sigma}_A)|\psi^-\rangle_{AB} \\ =\ &-tr_A\left(\rho_A(\vec{n}\cdot\vec{\sigma}_A)(\vec{m}\cdot\vec{\sigma}_A)\right) \quad \left(\rho_A = tr_B(|\psi^-\rangle_{AB}\langle\psi^-|) = \tfrac{1}{2}I_A\right) \\ =\ &-\vec{n}\cdot\vec{m} \end{aligned}$$



$|\psi^-\rangle = \frac{1}{\sqrt{2}}(|01\rangle - |10\rangle)$.

$\Rightarrow |W| = 2\sqrt{2} > 2$.

3. Experimental test of Bell's Inequality

Loophole $\begin{cases} \text{Nonlocality loophole (light cannot influence each other).} \\ \text{detection efficiency loophole. } \eta \ge 83\%. \\ \text{free-will loophole (randomness exists in nature?).} \end{cases}$ } 2016 closed. by diamond defect

Entanglement Based Quantum Communication

Entanglement Based Quantum ...

1. Quantum Teleportation

Alice $\longrightarrow$ Bob

$|\psi\rangle = C_0|0\rangle + C_1|1\rangle$
unknown

with only C.C. Alice need to know $C_0 . C_1$ $\Rightarrow$ infinite bits for infinite precision.

with only entanglement A —— B ($|00\rangle + |11\rangle$) cannot give information !

Quantum Teleportation = Entanglement + CC.

Protocol:

Alice. —— Bob

1) Bell measurement. 3• 1• —— 2• 2) CC.(outcomes) 3) recover

$|\phi^+\rangle_{12} = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$.

$|\psi\rangle_3 = a|0\rangle + b|1\rangle$

Start with

$|\psi\rangle_3 \otimes |\phi^+_{12}\rangle$

$= (a|0\rangle_3 + b|1\rangle_3) \otimes \frac{1}{\sqrt{2}}(|00\rangle_{12} + |11\rangle_{12})$.

$= \frac{1}{2}\left(|\phi^+\rangle_{31}|\psi\rangle_2 + |\phi^-\rangle_{31} Z_2|\psi\rangle_2 + |\psi^+\rangle_{31} X_2|\psi\rangle_2 + |\psi^-\rangle_{31} Z_2X_2|\psi\rangle_2 \right)$

where $|\psi_2\rangle = a|0\rangle_2 + b|1\rangle_2$.

Transfer: only two bits. info. $\Rightarrow$ Alice measure on Bell basis and tell Bob the outcome.

where $|\psi\rangle_3$ is unknown to Alice & Bob.

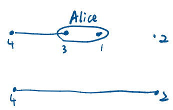
$$\left.\begin{cases} |\phi^+\rangle \Rightarrow I_2. \\ |\phi^-\rangle \Rightarrow Z_2 \\ |\psi^+\rangle \Rightarrow X_2 \\ |\psi^-\rangle \Rightarrow X_2Z_2 \end{cases}\right\} \Rightarrow |\psi\rangle_2 = a|0\rangle_2 + b|1\rangle_2$$

Remark: 1) Alice know nothing about $|\psi\rangle_3$.

so this does not contradicts with non clone.

2) We can let $\begin{cases} a = |0\rangle_4 \\ b = |1\rangle_4 \end{cases}$

$|\psi\rangle_{43} = \frac{1}{\sqrt{2}}(|00\rangle_{43} + |11\rangle_{43}) \Rightarrow |\psi\rangle_{42} = \frac{1}{\sqrt{2}}(|00\rangle_{42} + |11\rangle_{42})$



Entanglement Swapping

2. Quantum Key Distribution.

Alice —— Bob.

$|\phi^+\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle) = \frac{1}{\sqrt{2}}(|++\rangle + |--\rangle)$

	A	B.
Z basis	0	0
	1	1
X basis	+	+
	−	−

1) Alice & Bob. randomly measure in Z or X. basis and keep the outcomes as the key if they find they use the same basis. through C.C. (EPR protocol. / Ekert protocol)

2) Device - Indep. QKD. (Best Secure protocol)
Measure CHSH inequality on a randomly chosen subset of samples.
If violated, measure in the same basis another random sample and keep the outcomes as the shared key.

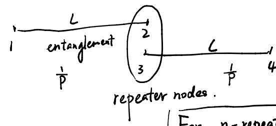
3. Quantum Repeaters.

Long distances communication:

Alice $\xrightarrow{\text{(light)}}$ Bob.

Exponential Decay $e^{-\frac{x}{L}}$ ← 10~20 km $\Rightarrow$ Need a repeater to amplify periodicly.

Quantum Nonclonning : no amplification of quantum signals



$\Rightarrow$ 1 —— 2L, entangment —— 4

Bell measurement

$\sim \frac{2}{p}$ with repeater

$\sim \frac{1}{p^2}$ without.

For n-repeaters

$$\boxed{\frac{1}{p^n} \longrightarrow \frac{n}{p}}$$

For realization:

1) Quantum memory to save entanglement
2) Flying qubits to establish entanglement between memories.

DLCZ schemes (not practical yet)