Quantum Dynamics

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Quantum Dynamic

1. Qubit Systems

- 2-D Hamiltonian $H = \hbar(\alpha I + \frac{\omega}{2}\vec{r}\cdot\vec{\sigma})$

  Assuming $\alpha, \omega, \vec{r}$ time Independent.

- Evolution:

$$U(t) = e^{-i\hat{H}t/\hbar}$$

$$= \underbrace{e^{-i\alpha t}}_{\text{global phase, drop.}} \underbrace{\left(I\cos\frac{\omega t}{2} - i\vec{r}\cdot\vec{\sigma}\sin\frac{\omega t}{2}\right)}_{\text{oscillation. (Rabi).}}$$

Examples.

1). If $\vec{r}\cdot\vec{\sigma} = \sigma_y$

$$U(t) = \cos\frac{\omega t}{2}\cdot\begin{pmatrix}1&0\\0&1\end{pmatrix} - i\begin{pmatrix}0&-i\\i&0\end{pmatrix}\sin\frac{\omega t}{2}$$

$$= \begin{pmatrix}\cos\frac{\omega t}{2} & -\sin\frac{\omega t}{2}\\ \sin\frac{\omega t}{2} & \cos\frac{\omega t}{2}\end{pmatrix}.$$

$$|\psi(t)\rangle = U(t)|\psi(0)\rangle = \cos\frac{\omega t}{2}|0\rangle - \sin\frac{\omega t}{2}|1\rangle.$$

when $\omega t = \frac{\pi}{2} \Rightarrow \begin{cases}|0\rangle \to \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)\\ |1\rangle \to \frac{1}{\sqrt{2}}(|1\rangle + |0\rangle)\end{cases}$

called $R_y(\frac{\pi}{2})$. ← gate operation.

Similarly, $\omega t = \pi$. (π-pulse), $U(t) = \begin{pmatrix}0&-1\\1&0\end{pmatrix}$.

2. Atomic Clocks. and Ramsey Method.

$|0\rangle$ ↑ $\Delta E = \hbar\omega_0$ ↓ $|1\rangle$

$|0\rangle \to |0\rangle e^{-\frac{i\omega t}{2}}$

$|1\rangle \to |1\rangle e^{-\frac{i\omega t}{2}}$.

$$\hat{H} = \frac{\hbar\omega_0}{2}\sigma_z.$$

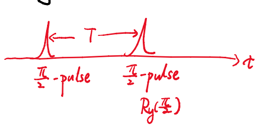
2). If $\vec{r}\cdot\vec{\sigma} = \sigma_z$.

$$U(t) = \cos\frac{\omega t}{2}\begin{pmatrix}1&0\\0&1\end{pmatrix} - i\sin\frac{\omega t}{2}\begin{pmatrix}1&0\\0&-1\end{pmatrix}.$$

$$= \begin{pmatrix}e^{-\frac{i\omega t}{2}} & 0\\ 0 & e^{i\omega t/2}\end{pmatrix}$$

$\begin{cases}|0\rangle \to |0\rangle e^{-i\frac{\omega t}{2}}\\ |1\rangle \to |1\rangle e^{i\omega t/2}\end{cases}$ a relative phase shift. denote by $R_z(\theta)$

Ramsey Method.



Initially at $|1\rangle$ state.

$$|1\rangle \xrightarrow{R_y(\frac{\pi}{2})} \frac{1}{\sqrt{2}}(|1\rangle + |0\rangle) \xrightarrow[\text{(free evolution)}]{R_z(\omega_0 T)} \frac{1}{\sqrt{2}}\left(|1\rangle e^{i\frac{\omega_0 T}{2}} + |0\rangle e^{-i\frac{\omega_0 T}{2}}\right) \xrightarrow{R_y(\frac{\pi}{2})} |0\rangle\cos\frac{\omega_0 T}{2} + i|1\rangle\sin\frac{\omega_0 T}{2}.$$

Measure prob. of $|1\rangle$:

$$P_1(t) = \sin^2\frac{\omega_0 t}{2}. \Rightarrow \text{atomic clocks.}$$

3. Multi-pertite systems

$$\underset{\text{exchange}}{S}|\psi\rangle_{12} = \pm|\psi_{12}\rangle \Rightarrow \begin{cases}+1 : \text{Bosons} \Rightarrow \text{symmetric}\\ -1 : \text{Fermions} \Rightarrow \text{anti-symmetric}\end{cases}$$

distinguish ⟶ partites ⟶ entanglement

The Hibert Space can be decomposed as

$$\mathcal{H} = \mathcal{H}_1 \otimes \mathcal{H}_2 \otimes \cdots \otimes \mathcal{H}_n$$

↳ tensor product

$\dim\mathcal{H} = \dim\mathcal{H}_1 \times \dim\mathcal{H}_2 \times \cdots \dim\mathcal{H}_n$

Ex.

1 qubit $|0\rangle$ $|1\rangle$

2 qubits $|00\rangle$ $|01\rangle$ $|10\rangle$ $|11\rangle$

......

n qubits $|000\cdots0\rangle$, $\cdots$ $|11\cdots1\rangle$ $\dim = 2^n$ tensor product state.

... Many-particle $\xrightarrow{\text{distinguish by modes}}$ multi-partite systems

n qubits $|00\cdots0\rangle, \cdots |11\cdots1\rangle$ $\dim = 2^n$ tensor product space

Remark: 1) Many-particle $\xrightarrow{\text{distinguish by modes}}$ multi-partite systems

2) Consider a special state

$$|\psi\rangle_{12\cdots n} = |\psi_1\rangle_1 \otimes |\psi_2\rangle_2 \otimes \cdots \otimes |\psi_n\rangle_n$$

degree of freedom: $2n$. product state

$2n \neq 2^n$ because product state does not contain entanglement.
but tensor product does.

mean field approx.

4. Density Matrix (Operator)

$$\rho \overset{\text{def}}{=} p_1 |\psi_1\rangle\langle\psi_1| + p_2 |\psi_2\rangle\langle\psi_2|$$

$$\langle A\rangle = p_1 \langle\psi_1|A|\psi_1\rangle + p_2 \langle\psi_2|A|\psi_2\rangle$$
$$= tr(\rho A)$$

Properties:

$$\begin{cases} tr(|\psi_1\rangle\langle\psi_2|) = \langle\psi_2|\psi_1\rangle \\ tr(A|\psi_1\rangle\langle\psi_2|) = \langle\psi_2|A|\psi_1\rangle \\ tr(A) = \sum_i \langle u_i|A|u_i\rangle \end{cases}$$

– Reduced state interpretation

Open systems 1 System — 2. environment $|\psi_{12}\rangle$

$$\langle A_1\rangle = \langle\psi_{12}|A_1|\psi_{12}\rangle$$

$tr_{1,2} = tr_1 tr_2$

$$= tr_{1,2}(A_1|\psi_{12}\rangle\langle\psi_{12}|)$$
$$= tr_1(A_1 \underbrace{tr_2(|\psi\rangle_{12}\langle\psi|)}_{\text{partial state } \rho_1})$$
$$= tr_1(A_1\rho_1)$$

$\rho_1 = tr_2(|\psi\rangle_{12}\langle\psi|)$
reduced state for system 1.

Ex. Calculate partial trace

$$|\psi_{12}\rangle = c_0|00\rangle_{12} + c_1|11\rangle_{12}$$

$$\rho_1 = tr_2(|\psi\rangle_{12}\langle\psi|)$$
$$= tr_2[\,|c_0|^2 |00\rangle_{12}\langle 00| + c_0c_1^* |00\rangle_{12}\langle 11| +$$
$$c_1c_0^* |11\rangle_{12}\langle 00| + |c_1|^2 |11\rangle_{12}\langle 11|\,]$$

$$tr_2(|00\rangle_{12}\langle 00|) = |0\rangle_1 \otimes tr_2(|0\rangle_2\langle 0|) \otimes_1\langle 0|$$
$$= |0\rangle_1\langle 0|_1$$

$$\Rightarrow \rho_1 = |c_0|^2 |0\rangle_1\langle 0| + |c_1|^2 |1\rangle\langle 1|$$

– Characterization theorem:

Any hermitian operator ρ is a density operator,
iff 1) $tr\,\rho = 1$. 2) $\rho \geq 0$. (any eigenvalues are nonnegative)

Proof. $\Rightarrow$ $\rho = \sum_i p_i |\psi_i\rangle\langle\psi_i|$

For any $|\psi\rangle$, $\langle\phi|\rho|\phi\rangle = \sum_i p_i \langle\phi|\psi_i\rangle\langle\psi_i|\phi\rangle$
$$= \sum_i p_i |\langle\phi|\psi_i\rangle|^2 \geq 0$$

$$tr(\rho) = \sum_i p_i \langle\psi_i|\psi_i\rangle = 1.$$

$\Leftarrow$ If a hermitian ρ $\begin{cases} tr\,\rho = 1 \\ \rho \geq 0. \end{cases}$

Spectral decomposition. $\rho = \sum_i \lambda_i |\lambda_i\rangle\langle\lambda_i|$.

$$\Rightarrow \begin{cases} tr\rho = \sum_i \lambda_i = 1. \\ \lambda_i \geq 0 \end{cases} \Rightarrow \lambda_i \text{ is a prob.}$$

$\{\lambda_i, |\lambda_i\rangle\}$: ensemble decomposition of ρ

Ex. $\rho = \frac{1}{2}I = \frac{1}{2}(|0\rangle\langle 0| + |1\rangle\langle 1|)$ $\Rightarrow$ Ensemble decomposition is not unique.
$$= \frac{1}{2}(|+\rangle\langle +| + |-\rangle\langle -|)$$

— Pure state criterion

$tr(\rho^2) \le 1$ and $tr(\rho^2) = 1$ iff ρ is a pure state

Proof. ρ has spectral decomposition.

$$\rho = \sum_i \lambda_i |\lambda_i\rangle\langle\lambda_i| .$$

$$tr(\rho^2) = \sum_i \lambda_i^2 \le \sum_i \lambda_i = 1 .$$

Equality holds iff $\exists \lambda_i = 1$ and the other $\lambda_i = 0$

$\Rightarrow \rho = |\psi_{i_0}\rangle\langle\psi_{i_0}| \Rightarrow$ a pure state.

5. Quantum entanglement & von-Neumann Entropy

Def. For pure state. $|\psi\rangle_{AB}$ of parties A and B.

If $|\psi\rangle_{AB} \ne |\phi_A\rangle_A \otimes |\phi_B\rangle_B$ for any ϕ_A, ϕ_B.

$|\psi\rangle_{AB}$ is an entangled state.

Ex. $|00\rangle_{AB}$

$(|0\rangle + |1\rangle)_A |0\rangle_B$ — product

$(|00\rangle + |01\rangle + |10\rangle + |11\rangle)_{AB}$

$|00\rangle_{AB} + |11\rangle_{AB}$ —— entangled.

How to quantify entanglement ?

$|\psi_{AB}\rangle$ — product $\Rightarrow \rho_A = |\phi_A\rangle\langle\phi_A|$ pure

$|\psi_{AB}\rangle$ — Entangled $\Rightarrow \rho_A$ is mixed

$\boxed{\text{Entanglement of } |\psi\rangle_{AB} \Leftrightarrow \text{Mixedness of } \rho_A !}$

Ex. $|00\rangle + |11\rangle \Rightarrow \rho_A = \frac{1}{2}(|0\rangle\langle 0| + |1\rangle\langle 1|)$

Def. von-Neumann entropy of ρ

$$S(\rho) \overset{def}{=} -tr(\rho \log \rho)$$

$$= -\sum_i \lambda_i \log \lambda_i \quad (\lambda_i \text{ are eigenvalues of } \rho).$$

$\Rightarrow E(|\psi\rangle_{AB}) \overset{def}{=} S(\rho_A)$ for pure state.

Ex. $|\psi\rangle_{AB} = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)_{AB}$, $\rho_A = \frac{1}{2}(|0\rangle\langle 0| + |1\rangle\langle 1|)$. $S(\rho) = 1$.

Properties :

1) $S(U^\dagger \rho U) = S(\rho)$

2). $S(|\psi\rangle\langle\psi|) = 0$

3) $S(\lambda_1 \rho_1 + \lambda_2 \rho_2) \ge \lambda_1 S(\rho_1) + \lambda_2 S(\rho_2)$ Convex of log function.

$(\lambda_1 + \lambda_2 = 1)$

4) Subadditivity.

$S(\rho_{AB}) \le S(\rho_A) + S(\rho_B)$

$\rho_A = tr_B(\rho_{AB})$

Equality holds iff $\rho_{AB} = \rho_A \otimes \rho_B$

Equality holds iff $\rho_{AB} = \rho_A \otimes \rho_B$

5). Strong Subadditivity.

$$S(\rho_{ABC}) + S(\rho_B) \le S(\rho_{AB}) + S(\rho_{BC})$$

6. Schmidt decomposition.

bi-partite (A.B)

$|i\rangle_A$, $|\mu\rangle_B$

$$\Rightarrow |\psi\rangle_{AB} = \sum_{i,\mu} a_{i\mu} |i\rangle_A |\mu\rangle_B$$

<u>Thm</u> $\exists$ basis $\{|i\rangle_A\}$ s.t.

$$|\psi\rangle_{AB} = \sum_i \sqrt{p_i}\, |i\rangle_A |\tilde{i}\rangle_B \qquad |\tilde{i}\rangle_B \equiv \sum_\mu a_{i\mu} |\mu_B\rangle$$

orthonormal

<u>Proof</u>. The reduced state $\rho_A = tr_B(|\psi\rangle_{AB}\langle\psi|)$

$$= tr_B\Big(\sum_{ij} |i\rangle_A |\tilde{i}\rangle_B\, {}_B\langle\tilde{j}|\, {}_A\langle j|\Big)$$

$$= \sum_{ij} |i\rangle_A\langle j| \,\big(tr_B\, |\tilde{i}\rangle_B\langle\tilde{j}|\big)$$

$$= \sum_{ij} |i\rangle_A\langle j| \,\big({}_B\langle\tilde{j}|\tilde{i}\rangle_B\big)$$

Choose $|i\rangle_A$ to be eigenstates of $\rho_A = \sum_i p_i |i\rangle_A\langle i|$

Therefore, ${}_B\langle\tilde{j}|\tilde{i}\rangle_B = p_i \delta_{ij}$

Let $|\mu_i\rangle \equiv \frac{1}{\sqrt{p_i}} |\tilde{i}\rangle_B \Rightarrow {}_B\langle\mu_i|\mu_j\rangle_B = \delta_{ij}$

$$\Rightarrow |\psi\rangle_{AB} = \sum_i \sqrt{p_i}\, |i\rangle_A |\mu_i\rangle_B$$

Applications:

1) $\rho_B = \sum_i p_i |\mu_i\rangle_B\langle\mu_i|$

$\Rightarrow$ eigenvalues of ρ_A, ρ_B are p_i.

$$S(\rho_A) = S(\rho_B) = E(|\psi\rangle_{AB})$$

2) Purification of density operator.

Any mixed state ρ_A can be written as a reduced state with $\rho_A = tr_B(|\psi\rangle_{AB}\langle\psi|)$.

where $|\psi\rangle_{AB}$ is a purification of ρ_A.

and $\dim(\mathcal{H}_B) \le \dim(\mathcal{H}_A)$.

<u>Proof</u> $\rho_A = p_i |i\rangle_A\langle i|$

$$\Rightarrow |\psi\rangle_{AB} = \sum_i \sqrt{p_i}\, |i\rangle_A |\mu_i\rangle_B$$

a purification of ρ_A

Generalized evolution.

	State	evolution	measurement
Closed system	$\vert\psi\rangle$	U	Projector.
Open system	ρ	$\hat{\$}$ (superoperator)	POVM

<u>Def</u> Operator: a map from state to state

Superoperator: a map from operator to operator.

1. Properties of $\hat{\$}$.

   1) Linear $\hat{\$}(\lambda_1\rho_1 + \lambda_2\rho_2) = \lambda_1 \hat{\$}(\rho_1) + \lambda_2 \hat{\$}(\rho_2)$.

   linearity on density. different linearity on state

   $U(c_1|\psi_1\rangle + c_2|\psi_2\rangle) = c_1 U|\psi_1\rangle + c_2 U|\psi_2\rangle$

   2) Trace preserving. $(tr(\rho) = 1 \Rightarrow tr(\hat{\$}(\rho)) = 1)$

   physical requirement.

1) Linear $\$(\lambda_1\rho_1+\lambda_2\rho_2) = \lambda_1 \$(\rho_1) + \ldots$

2) Trace preserving. $(\mathrm{tr}(\rho)=1 \Rightarrow \mathrm{tr}(\hat{\$}(\rho))=1)$

→ physical requirement.

3). Hermitian preserving ρ is Hermitian $\Rightarrow \hat{\$}(\rho)$ Hermitian.

4) Positive $\hat{\$}(\rho)$ nonnegative if ρ nonnegative.

4') If $\hat{\$}_A \otimes \hat{I}_B$ is positive for any extension of B, $\hat{\$}_A$ is called completely positive.

Ex.

$\hat{T}: \rho \to \rho^T$

$\Rightarrow$ positive. superoperator

$\hat{T}_A \otimes \hat{I}_B$: partial transpose.

Apply to $|\Phi\rangle_{AB} = \sum_{i=1}^{N} |i\rangle_A \otimes |i\rangle_B$, ${}_{AB}\langle\Phi|\Phi\rangle_{AB} = N$ normalize.

$\rho_{AB} = |\Phi_{AB}\rangle\langle\Phi|$

$\hat{T}_A \otimes \hat{I}_B(\rho_{AB}) = \hat{T}_A \otimes \hat{I}_B\left[\sum_{i,j} |i\rangle_A\langle j| \otimes |i\rangle_B\langle j|\right]$

$= \sum_{i,j} |j\rangle_A\langle i| \otimes |i\rangle_B\langle j|$

For any state $|\phi_1\rangle_A \otimes |\phi_2\rangle_B = \sum_i a_i|i\rangle_A \otimes \sum_j b_i|j\rangle_A$

Then $\rho'_{AB}(|\phi_1\rangle_A \otimes |\phi_2\rangle_B) = \sum_{i,j} a_i b_j |j\rangle_A |i\rangle_B = |\phi_2\rangle_A \otimes |\phi_1\rangle_B$

↳ swap operator ρ' : on an anti-symmetric $|\psi\rangle_{AB}$

$\Rightarrow$ ρ' is not positive $\Rightarrow$ $\$ = \hat{T}$ is not completely positive.

2. Thm (Kraus representation).

Any $\hat{\$}$ satisfies (1) (2) (3) (4)

has an operator-sum representation:

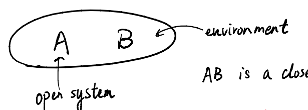
$$\hat{\$}(\rho_A) = \sum_\mu M_\mu \rho_A M_\mu^\dagger$$

← generalization of Unitary : $\rho \to u\rho u^\dagger$.

where $\sum_\mu M_\mu^\dagger M_\mu = I_A$

M_μ: Kraus operator.

An Informal Proof



AB is a closed system.

$\rho_A \xrightarrow{\text{extension}} \rho_A \otimes \rho_B$

no entanglement initially

→ This is not a trivial assumption. that the correlation of A and the universe can be neglected.

W.L.G. write $\rho_B = |E\rangle_B\langle E|$ (otherwise double its dimension by purification)

After evolution. $\rho'_{AB} = U_{AB}(\rho_A \otimes |E\rangle_B\langle E|)U^\dagger_{AB}$.

$\rho'_A = \mathrm{tr}_B(\rho'_{AB})$ ← $I_B = \sum_\mu |\mu_B\rangle\langle\mu_B|$

$= \sum_\mu {}_B\langle\mu| U_{AB}(\rho_A \otimes |E\rangle_B\langle E|) \underbrace{U^\dagger_{AB}|\mu\rangle_B}_{M_\mu^\dagger}$

Define.

$M_\mu = {}_B\langle\mu| U_{AB}|E\rangle_B$

$\Rightarrow \rho'_A = \sum_\mu M_\mu \rho_A M_\mu^\dagger$

$\sum_\mu M_\mu^\dagger M_\mu = \sum_\mu {}_B\langle E|U^\dagger_{AB}|\mu\rangle_B \langle\mu|U_{AB}|E\rangle_B$

$= {}_B\langle E| I_{AB}|E\rangle_B = I_A$

3. Master Equation.

$$\rho(t+\delta t) = \hat{\$}(\rho(t)) = \sum_\mu M_\mu \rho(t) M_\mu^\dagger$$

$$\rho(t) \xrightarrow{\delta(t)} \rho(t+\delta t)$$

No initial entanglement at each time step

↳ $\boxed{\text{Markovian approximation}}$

$\Rightarrow \rho(t+\delta t) = \rho(t) + \delta t \cdot \dot{\rho}$

$= \sum_\mu M_\mu \rho M_\mu^\dagger$

$\Rightarrow$ Let $M_0 = I + \delta t(K - iH)$ (Re, Im.)

other terms close to 0.

$$\Rightarrow \rho(t+\delta t) = \rho(t) + \delta t \cdot \dot{\rho} = \sum_\mu M_\mu \rho M_\mu^\dagger$$

$\Rightarrow$ Let $M_0 = I + \delta t\,(\overset{\text{Re}}{K} - i\overset{\text{Im.}}{H})$

Other terms close to 0.

$$\sum_\mu M^\dagger M = \underbrace{M_0^\dagger M_0}_{I+2\delta t \cdot K} + \sum_{\mu\neq 0} M_\mu^\dagger M_\mu = I.$$

$\Rightarrow M_\mu \propto \sqrt{\delta t}$. Let $M_\mu = L_\mu \sqrt{\delta t}$

Thus, $\sum_\mu L_\mu^\dagger L_\mu + 2K = 0$

$$\Rightarrow \dot{\rho} = \underbrace{-i[H,\rho]}_{\text{Hamiltonian for closed system}} + \underbrace{\sum_\mu [L_\mu \rho L_\mu^\dagger - \tfrac{1}{2} L_\mu^\dagger L_\mu \rho - \tfrac{1}{2}\rho L_\mu^\dagger L_\mu]}_{\text{Lindbiad form (for open system)}}$$

L_μ : Jump operator / decay channel

4. Examples of Superoperator evolution.

$$\rho \xrightarrow{\hat{\$}} \hat{\$}(\rho).$$

(1) Depolarization Channel for a qubit

$1-p$ no error

p:
- $\frac{p}{3}$ a bit flip error $\sigma_x = X$ $|0\rangle \leftrightarrow |1\rangle$
- $\frac{p}{3}$ a phase flip error $\sigma_z = Z$ $|+\rangle \leftrightarrow |-\rangle$
- $\frac{p}{3}$ a bit-phase ... $\sigma_y = Y = iXZ$

$$\Rightarrow M_0 = \sqrt{1-p}\, I \quad M_1 = \sqrt{\frac{p}{3}}\, X, \quad M_2 = \sqrt{\frac{p}{3}}\, Y. \quad M_3 = \sqrt{\frac{p}{3}}\, Z$$

$$\hat{\$}(\rho) = (1-p)\rho + \frac{p}{3}(X\rho X + Y\rho Y + Z\rho Z).$$

$\rho = \frac{1}{2}(I + \vec{n}\cdot\vec{\sigma})$ $\quad \|\cdot\|$

$\downarrow$

$\rho' = \frac{1}{2}(I + \vec{n}'\cdot\vec{\sigma})$

$\vec{n}' = (1-\frac{4}{3}p)\vec{n}$ $\Rightarrow$ $\|\vec{n}\|$ is shrinking!

$p = \frac{3}{4}$, maximum depolarization.

$$\dot{\rho} = \gamma\,(X\rho X + Y\rho Y + Z\rho Z \underbrace{-3\rho}_{-\frac{1}{2}x^2\rho - \frac{1}{2}\rho x^2 = -\rho})$$

(2) Phase-damping channel

- $1-p$ no error
- p $\sigma_z = Z$ error.

Kraus operator $M_0 = \sqrt{1-p}\, I$ $\quad M_1 = \sqrt{p}\, Z$

$$\$(\rho) = (1-p)\rho + pZ\rho Z = \begin{pmatrix} \rho_{00} & (1-2p)\rho_{01} \\ (1-2p)\rho_{10} & \rho_{11} \end{pmatrix}$$

$$\Rightarrow \dot{\rho} = \gamma\,(Z\rho Z - \rho)$$

(3) Amplitude damping channel $\Rightarrow$ purify a system ($t\to\infty$, to ground state).

$|1\rangle$

$|0\rangle$

$$\begin{cases} |0\rangle|E\rangle = |0\rangle|E\rangle \\ |1\rangle|E\rangle = \sqrt{1-p}\,|1\rangle|E\rangle + \sqrt{p}\,|0\rangle|F'\rangle \end{cases}$$

Kraus operator. $M_0 = \begin{pmatrix} 1 & 0 \\ 0 & \sqrt{1-p} \end{pmatrix}$ $\quad M_1 = \begin{pmatrix} 0 & \sqrt{p} \\ 0 & 0 \end{pmatrix} \propto |0\rangle\langle 1|$

$$\rho \to \hat{\$}(\rho) = \begin{pmatrix} \rho_{00} + p\rho_{11} & \sqrt{1-p}\,\rho_{01} \\ \underset{\approx 1-\frac{p}{2} = 1-\frac{t}{T_2}}{\sqrt{1-p}}\,\rho_{10} & \underset{= 1-\frac{t}{T_1}}{(1-p)}\,\rho_{11} \end{pmatrix}$$

T_1: life time of excited state

$T_2 = 2T_1$: coherence time

$$\Rightarrow \dot{\rho} = \gamma\,(\sigma_- \rho\, \sigma_+ - \tfrac{1}{2}\sigma_+\sigma_- \rho - \tfrac{1}{2}\rho\,\sigma_+\sigma_-)$$

$\sigma_- = |0\rangle\langle 1|$, $\sigma_+ = |1\rangle\langle 0|$.

Generalize Measurement

1. POVM (positive operator valued measurement)

Review : von-Neumann Projector $\{E_i\}$

States after a POVM : not uniquely determined by $\{F_i\}$

Example : Bell measurement

1. POVM (positive operator valued measurement)

Review : von-Neumann Projector $\{E_i\}$

$$|\psi\rangle \longrightarrow E_i|\psi\rangle$$

$$\rho \longrightarrow E_i \rho E_i$$

Extension to Composite systems

(A) B $\quad \rho_{AB} = \rho_A \otimes \rho_B$ (no entanglement at the beginning)

Prob. to get i : $P_i = tr_{AB}(\rho_{AB} E_i) = tr_A[\rho_A \underbrace{tr_B(\rho_B E_i)}_{F_i}]$

$$\Rightarrow P_i = tr_A(\rho_A F_i)$$

F_i : 1) Hermitian $\quad F_i^+ = F_i$

2) Positive $\quad {}_A\langle\phi|F_i|\phi\rangle_A$

3) Complete $\quad \sum_i F_i = I_A$

Example : Bell measurement

$$|\Phi^\pm\rangle = \frac{1}{\sqrt{2}}(|01\rangle \pm |10\rangle)$$

$$|\Psi^\mp\rangle = \frac{1}{\sqrt{2}}(|00\rangle \pm |11\rangle)$$

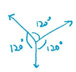
$E_1, E_2, E_3, E_4 \rightarrow$ Projection to Bell states

2. Inverse Thm. (Neumark)

Any POVM $\{F_i\}$ in $\mathcal{H}_A$ can be implemented with a projection measurement in $\mathcal{H}_A \otimes \mathcal{H}_B$ with same probability outcome. i.e.

$$tr(F_i \rho_A) = P_i = tr_B(E_i \rho_A \otimes \rho_B).$$

Ex. a POVM for a qubit



$\vec{n}_1 + \vec{n}_2 + \vec{n}_3 = 0$

Def. $\quad F_i = \frac{1}{3}(I + \vec{n}_i \cdot \vec{\sigma}) \longrightarrow$ check : Hermitian, positive, completeness

$\Rightarrow$ A initial state :

$$\rho = \frac{1}{2}(I + \vec{r}\cdot\vec{\sigma}) \Rightarrow P(F_i) = tr(\rho F_i) = \frac{1}{3} + \frac{1}{6} tr(\vec{n}_i\cdot\vec{\sigma}\,\vec{r}\cdot\vec{\sigma}) \qquad \begin{cases} tr\,\vec{\sigma} = 0 \\ tr(\sigma_i\sigma_j) = 2\delta_{ij} \end{cases}$$

$$= \frac{1}{3}(1 + \vec{n}_i \cdot \vec{r})$$

愿星光照耀你们前行的道路.