Quantum System

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1. Finite - dim quantum systems

Hilbert Space has a finite dim $\rightarrow$ dim = 2 $\Rightarrow$ qubit system

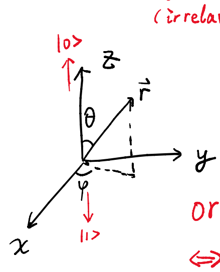
basis vectors $|e_0\rangle, |e_1\rangle, \cdots, |e_{d-1}\rangle$

Examples:
1) Spin-$\frac{1}{2}$ particles. $(|\uparrow\rangle, |\downarrow\rangle)$
2) Two-level atoms (ions) $(|1\rangle, |0\rangle)$
3) Photon polarization $(|H\rangle, |V\rangle)$
4) Quantum dots.
5) Superconducting qubits (Josephson junctions)

2. Description of qubit systems.

Basic vectors $|0\rangle = \begin{pmatrix}1\\0\end{pmatrix}$, $|1\rangle = \begin{pmatrix}0\\1\end{pmatrix}$

$$|\psi\rangle = \underbrace{e^{i\gamma}}_{\substack{\text{global phase}\\\text{(irrelavent)}}} \left(\cos\frac{\theta}{2}|0\rangle + \underbrace{\sin\frac{\theta}{2}}_{\text{relative amplitude}} \underbrace{e^{i\varphi}}_{\text{relative phase}} |1\rangle\right)$$



$$\hat{\rho} = \frac{1}{2}(I + r\cdot\sigma)$$
$$\hat{\rho} = |\psi\rangle\langle\psi|$$

orthogonal state

$\Leftrightarrow$ vectors in opposite direction.

Operators: 2x2 Hermite matrix.

- Basis: $I = \begin{pmatrix}1&0\\0&1\end{pmatrix}$, $\sigma_x = \begin{pmatrix}0&1\\1&0\end{pmatrix}$, $\sigma_y = \begin{pmatrix}0&-i\\i&0\end{pmatrix}$, $\sigma_z = \begin{pmatrix}1&0\\0&-1\end{pmatrix}$. $\rightarrow$ Eigenvectors
  - x: $|+\rangle, |-\rangle$
  - y: $|L\rangle, |R\rangle$
  - z: $|\uparrow\rangle, |\downarrow\rangle$

$$\begin{cases} \sigma_i^2 = I \\ [\sigma_i, \sigma_j] \overset{\text{def}}{=} \sigma_i\sigma_j - \sigma_j\sigma_i = 2\varepsilon_{ijk}\, i\sigma_k \\ \{\sigma_i, \sigma_j\} \overset{\text{def}}{=} \sigma_i\sigma_j + \sigma_j\sigma_i = 0 \end{cases}$$

$$\hat{S} = \frac{\hbar}{2}\hat{\sigma}$$

3. Continuous-variable quantum systems.

$$\underbrace{\hat{x}}_{\text{operator}} = \int \underbrace{x}_{\text{eigenvalue}} \underbrace{|x\rangle\langle x|}_{\text{eigenvectors}}\, dx$$

$$\langle x|x'\rangle = \delta(x-x')$$
$$\int f(x)\,\delta(x-x')\,dx = f(x')$$

$|p\rangle$ in term of $|x\rangle$

$$|p\rangle \propto \int e^{ikx}|x\rangle$$
$$|p\rangle = \frac{1}{\sqrt{2\pi\hbar}}\int e^{ipx/\hbar}|x\rangle$$
$$\Rightarrow \langle x|p\rangle = \frac{1}{\sqrt{2\pi\hbar}} e^{ipx/\hbar}$$

4. Coordinate and momentum representation

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$|\psi\rangle \Rightarrow \psi(x) \overset{def}{=} \langle x|\psi\rangle$ (in coordinate-basis)

State $\Rightarrow \psi(p) \overset{def}{=} \langle p|\psi\rangle$

$$\psi(p) = \langle p|\psi\rangle = \int \langle p|x\rangle\langle x|\psi\rangle\, dx$$
$$= \frac{1}{\sqrt{2\pi\hbar}} \int \psi(x)\, e^{-ipx/\hbar} dx$$

$$\Rightarrow |p\rangle = \int p\, |p\rangle\langle p|\, dp$$
$$= \int p\, |x\rangle\langle x|p\rangle\langle p|x'\rangle\langle x'|\, dp\, dx' dx$$
$$= \int p\, |x\rangle\langle x'| \frac{e^{ip(x-x')/\hbar}}{2\pi\hbar}\, dp\, dx\, dx'$$
$$= \int |x\rangle\langle x'| \left(i\hbar \frac{d}{dx'}\right) \frac{e^{ip(x-x')/\hbar}}{2\pi\hbar}\, dp\, dx\, dx'$$
$$= \int |x\rangle \left(-i\hbar\frac{d}{dx'}\right)\langle x'| \cdot \delta(x-x')\, dx\, dx'$$
$$= \int |x\rangle p \langle x|\, dx$$

$$[\hat{x}, \hat{p}] = [\hat{x}, -i\hbar\frac{d}{dx}] = i\hbar$$

$$\Rightarrow \quad \hat{p} \Leftrightarrow -i\hbar\frac{d}{dx} \quad \text{in coordinate basis.}$$

5. Uncertainty relation

Operator A, B

$$\langle (\Delta A)^2 \rangle \langle (\Delta B)^2 \rangle \geq \frac{1}{4} |\langle [A,B] \rangle|^2$$

6. Schrödinger picture & Heisenberg Picture

Schrödinger: $\begin{cases} \text{state evolves} \\ \text{operators don't evolve} \end{cases}$

$$\Rightarrow i\hbar \frac{\partial|\psi\rangle}{\partial t} = H|\psi\rangle \qquad \Longrightarrow \hat{H} = \frac{\hat{p}^2}{2m} + V(\hat{x})$$

In coordinate basis:

$$i\hbar\frac{\partial}{\partial t}\langle x|\psi\rangle = \langle x|H|\psi\rangle$$

	State	Operator	physically observed property
SP	$\|\psi(t)\rangle_s = \mathcal{U}(t)\|\psi(0)\rangle$	$A_s = A$	$\langle A\rangle_s = {}_s\langle\psi(t)\|A_s\|\psi(t)\rangle_s$
			$\|\|$
HP	$\|\psi(t)\rangle_H = \|\psi(0)\rangle_H = \|\psi(0)\rangle_s$	$A_H(t) = \mathcal{U}^\dagger(t) A \mathcal{U}(t)$	$\langle A\rangle_H = {}_H\langle\psi(t)\|A_H(t)\|\psi(t)\rangle_H$

$$i\hbar\frac{\partial|\psi(t)\rangle}{\partial t} = i\hbar\frac{\partial\mathcal{U}(t)}{\partial t}|\psi(0)\rangle = H|\psi(t)\rangle$$
$$= H\,\mathcal{U}(t)|\psi(0)\rangle$$

$$\Rightarrow i\hbar \frac{\partial U(t)}{\partial t} = H \cdot U(t)$$

$$i\hbar \frac{dA_H}{dt} = i\hbar \left[\frac{\partial U^+(t)}{\partial t} A U + U^+ \frac{\partial A}{\partial t} U + U^+ A \frac{\partial U}{\partial t} \right] \qquad i\hbar \frac{\partial U}{\partial t} = HU$$

$$= i\hbar \frac{\partial A_H}{\partial t} + A_H H_H - H_H A_H$$

$$= i\hbar \frac{\partial A_H}{\partial t} + [A_H, H_H]$$

If A_S is not explicitly t-dependent

$$\Rightarrow \frac{\partial A_H}{\partial t} = 0 \qquad \frac{dA_H}{dt} = [A_H, H_H]$$

7. Harmonic oscillator.

$$\hat{H} = \frac{p^2}{2m} + \frac{1}{2} m\omega^2 \hat{x}^2$$

$$\Rightarrow \left.\begin{array}{l} i\hbar \frac{d\hat{x}}{dt} = [\hat{x}, \hat{H}] = \frac{\hat{p}}{m} \\ i\hbar \frac{d\hat{p}}{dt} = [\hat{p}, \hat{H}] = -m\omega^2 \hat{x} \end{array}\right\} \quad \frac{d^2\hat{x}}{dx} = -\omega^2 \hat{x}$$

$$\begin{cases} \hat{x}(t) = \hat{x}(0) \cos\omega t + \frac{\hat{p}(0)}{m\omega} \sin\omega t \\ \hat{p}(t) = \hat{p}(0) \cos\omega t - m\omega \hat{x}(0) \sin\omega t \end{cases}$$

Assume $\langle \hat{x}(0) \hat{p}(0) + \hat{p}(0) \hat{x}(0) \rangle = 0$

$$m\omega \langle \Delta \hat{x}(0) \rangle^2 = \frac{1}{m\omega} \langle (\Delta \hat{p}(0))^2 \rangle = \frac{\hbar}{2}$$ (at minimum certainty state)

$$\Rightarrow \langle \Delta(\hat{x}(t))^2 \rangle = \langle (\Delta \hat{x}(0))^2 \rangle$$

$$\langle \Delta \hat{p}(t)^2 \rangle = \langle (\Delta \hat{p}(0))^2 \rangle$$

$\longleftarrow$ coherent state.

Annihilation operator $\hat{a} = \sqrt{\frac{m\omega}{2\hbar}} \hat{x} + i \frac{\hat{p}}{\sqrt{2m\hbar\omega}}$

Creation operator $\hat{a}^+ = \sqrt{\frac{m\omega}{2\hbar}} \hat{x} - i \frac{\hat{p}}{\sqrt{2m\hbar\omega}}$

1) $[\hat{a}, \hat{a}^+] = 1$

2) $\hat{H} = \hbar\omega (\hat{a}^+ \hat{a} + \frac{1}{2})$

3) Define $\hat{n}$ (number operator) $\stackrel{def}{=} \hat{a}^+ \hat{a}$

$$\hat{n} |\psi_n\rangle = n |\psi_n\rangle$$

$$\begin{cases} |\psi_n^+\rangle \stackrel{def}{=} \hat{a}^+ |\psi_n\rangle \\ |\psi_n^-\rangle \stackrel{def}{=} \hat{a} |\psi_n\rangle \end{cases} \qquad \begin{cases} \hat{n} |\psi_n^+\rangle = (n+1) |\psi_n\rangle \\ \hat{n} |\psi_n^-\rangle = (n-1) |\psi_n\rangle \end{cases}$$

4) $\langle\psi|\hat{n}|\psi\rangle \geq 0$

minimum eigenvalue $\Rightarrow$ 0.

Therefore eigenspectrum of $\hat{n}$: 0, 1, 2. ⋯ .

$\hat{n}|n\rangle = n|n\rangle$

$n=0 \Rightarrow a^{\dagger}a|0\rangle = 0.$ Vacuum state.

$\hookrightarrow a|0\rangle = 0$

5) $\hat{H}|\psi_n\rangle = E_n|\psi_n\rangle$

$E_n = (n+\frac{1}{2})\hbar\omega$

$\langle x| \sqrt{\frac{m\omega}{2\hbar}}\hat{x} + i\frac{\hat{p}}{\sqrt{2m\hbar\omega}}|0\rangle = 0$

$\sqrt{\frac{m\omega}{2\hbar}}\hat{x} + \sqrt{\frac{\hbar}{2m\omega}}$