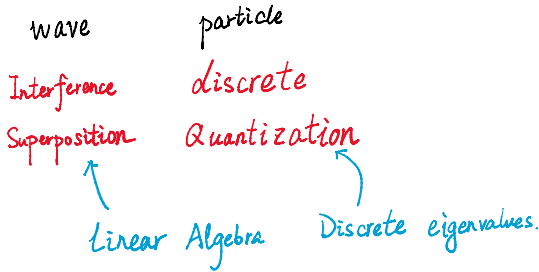


2.1. Formalism of Quantum mechanics.

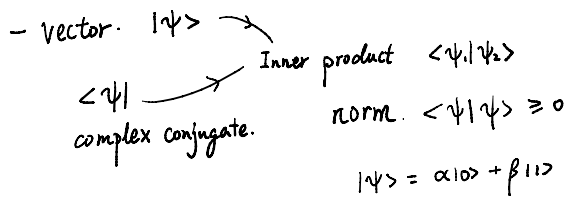
- States to describe systems
- evolution
- measurement / observables

1. Basic concept: wave-particle duality



2. States in Quantum mechanics.

Assumption 1: states \Leftrightarrow a vector in Hilbert Space



Hermitian:

$$A = A^\dagger$$

$$A^\dagger: \langle\psi_1|A|\psi_2\rangle = \langle\psi_2|A^\dagger|\psi_1\rangle^*$$

$$\begin{cases} \sum_i |\lambda_i\rangle\langle\lambda_i| = I \\ \langle\lambda_i|\lambda_j\rangle = \delta_{ij} \end{cases}$$

Spectral Decomposition

$$A = \sum_i \lambda_i A_i = \sum_i \lambda_i |\lambda_i\rangle\langle\lambda_i|$$

orthogonal projection (see LA 64)

real

outer product

$$|\psi\rangle = \sum_i |\lambda_i\rangle\langle\lambda_i|\psi\rangle$$

3. Observables in Quantum mechanics.

Assumption 2: Observables \Leftrightarrow a Hermitian operator. in Hilbert Space

$$|\psi\rangle \xrightarrow{A} A|\psi\rangle$$

↑ operators → a chosen basis → matrices.

| | | |
|------------|---------------|---|
| coordinate | X | \hat{X} |
| momentum | P | $\hat{P} = -i\hbar\nabla$ |
| Energy | $E = E(x, p)$ | $\hat{H}(\hat{x}, \hat{p}) = \frac{\hat{P}^2}{2m} + V(\hat{x})$ |

↑ Hamiltonian operator.

4. Evolution in Quantum mechanics.

$|\psi(t=t_0)\rangle \rightarrow |\psi(t=t_0+\Delta t)\rangle$

Assumption 3: Schrödinger Equation.

$$i\hbar \frac{\partial |\psi\rangle}{\partial t} = \hat{H}|\psi\rangle$$

if \hat{H} is time-independent.

$$A = \sum_i \lambda_i |\lambda_i\rangle\langle\lambda_i|$$

$$f(A) = \sum_i f(\lambda_i) |\lambda_i\rangle\langle\lambda_i|$$

$$\rightarrow -i\hat{H}t, \dots, e^{-i\frac{\hat{H}}{\hbar}t}$$

$$i\hbar \frac{\partial |\psi\rangle}{\partial t} = \hat{H} |\psi\rangle$$

if \hat{H} is time-independent.

$$|\psi(t)\rangle = e^{-i\hat{H}(t-t_0)/\hbar} |\psi(t_0)\rangle$$

Evolution operator $U(t, t_0)$

$$U^\dagger U = I \leftarrow \text{Unitary}$$

norm-preserving (see LA 6.5).

$$|\langle \psi(t) | \psi(t) \rangle| = \langle \psi(t_0) | U^\dagger U | \psi(t_0) \rangle = \langle \psi(t_0) | \psi(t_0) \rangle$$

$$e^{-i\hat{H}t/\hbar} |\psi\rangle = \langle \psi_{in} | \psi \rangle \cdot e^{-i\hat{H}t/\hbar} |\psi_{in}\rangle$$

5. Quantum no-cloning theorem

system A B

initial $|\psi\rangle \otimes |0\rangle \leftarrow$ blank state.

COPY $|\psi\rangle \otimes |\psi\rangle$

Tensor Product.

$$|\psi\rangle \otimes |\psi\rangle$$

Proof. If this process is possible

$\Rightarrow \exists$ unitary operator U :

$$\$. U(|\psi\rangle_A \otimes |0\rangle_B) = |\psi\rangle_A \otimes |\psi\rangle_B$$

This COPY should work for any $|\psi\rangle$.

$\Rightarrow |\psi_1\rangle, |\psi_2\rangle$ with $\langle \psi_1 | \psi_2 \rangle \neq 1$ or 0 .

$$\Rightarrow \begin{cases} U(|\psi_1\rangle_A |0\rangle_B) = |\psi_1\rangle_A |\psi_1\rangle_B \\ U(|\psi_2\rangle_A |0\rangle_B) = |\psi_2\rangle_A |\psi_2\rangle_B \end{cases} \xrightarrow{\text{(Inner product)}} \langle \psi_1 | \psi_2 \rangle_A \langle 0 | 0 \rangle_B = \langle \psi_1 | \psi_2 \rangle_A \langle \psi_1 | \psi_2 \rangle_B$$

Contradiction!

\Rightarrow Great for Security.

6. Measurement in Quantum mechanics.

Assumption 4. When measuring A under state $|\psi\rangle$.

the state is projected to an eigenstate $|\lambda_i\rangle$ of A with prob. $P_i = |\langle \lambda_i | \psi \rangle|^2$ and outcome $A = \lambda_i$

Define von Neumann projectors $P_i \stackrel{\text{def}}{=} |\lambda_i\rangle \langle \lambda_i|$

$$\begin{aligned} & \text{Initial } |\psi\rangle \\ & \text{Final (After Measurement)} \end{aligned} \Rightarrow \begin{cases} P_i \geq 0 \\ P_i^2 = P_i \\ P_i P_j = \delta_{ij} P_i \\ \sum_i P_i = I \end{cases}$$

$$P_i |\psi\rangle = \langle \lambda_i | \psi \rangle \cdot |\lambda_i\rangle$$