SVM

2020年10月12日 13:33

$\{x_i, y_i\}$. $y_i \in \{-1, 1\}$

Hard Margin (Perfectly Linear Separable).

- Margin: distance from the separator to the closest point

Margin length $\frac{1}{\|W\|_2}$

Goal: $y_i(W^T x_i - b) \geq 1$ (at the same time minimize $\|W\|_2$)

If data not perfectly linearly Separable ⇒ Allow mistakes

naive answer.

minimize $\|W\|_2 + \lambda \cdot \#$ mistakes

but indicator function is hard to optimize. (NP-hard)

$L_0 \xRightarrow{\text{relax}} L_1$

⇒ minimize $\|W\|_2 + \lambda \sum \xi_i$.

s.t. $\forall i$. $y_i(W_i^t x_i - b) \geq 1 - \xi_i$

$\xi_i \geq 0$. (slack variable)

Hinge loss: $\max\{0, 1 - ty\}$

($t = w_i^t x_i - b$ is the output).

Solve:

Primal — Dual

$$\min_w \|W\|_2 + \lambda \cdot \sum \xi_i$$

s.t. $\forall i$ $y_i(w^t X_i) \geq 1 - \xi_i$

$\xi_i \geq 0$

⇒

$$\max_a \sum_i a_i - \frac{1}{2} \sum_i \sum_j y_i y_j a_i a_j \langle x_i, x_j \rangle.$$

s.t. $\forall i$ $0 \leq a_i \leq \frac{1}{2n\lambda}$

$\sum_i y_i a_i = 0$

$$W = \sum_i a_i x_i y_i$$

Kernel Method

Input Space (not seperable) $x \longrightarrow \phi(x)$ feature space separable. (usually much higher dim)

With SVM:

$$\min_{w} \|W\|_2 + \lambda \cdot \sum \xi_i$$

s.t. $\forall i \quad y_i(w^t\phi(x_i)) \geq 1-\xi_i \quad \Rightarrow$

$\xi_i \geq 0$

$$\max_{a} \sum_i a_i - \frac{1}{2}\sum_i\sum_j y_i y_j a_i a_j \langle \phi(x_i), \phi(y_j) \rangle$$

s.t. $\forall i \quad 0 \leq a_i \leq \frac{1}{2n\lambda}$

$\sum_i y_i a_i = 0$

$$\Rightarrow w = \sum_i a_i y_i \phi(x_i)$$

$\phi(x_i)$ hard to compute

but $\langle \phi(x_i), \phi(x_j) \rangle = K(x_i, x_j)$ may be computed easily

↳ kernel trick.

predict:

$$w^t\phi(x) = \sum_i a_i y_i \langle \phi(x_i), \phi(x) \rangle$$

$\Rightarrow$ no need for computing $\phi(x)$

Mercer's Theorem

$$K = \begin{bmatrix} K(x_1, x_1) & K(x_1, x_2) & \cdots \\ K(x_2, x_1) & \ddots & \\ \vdots & & \ddots \end{bmatrix}$$

if K semi-definite for any $\{x_i\}$, then $\exists\, \phi$ such that K is a kernel for ϕ.

(We do not even need ϕ in computation!).