Traffic at Peak Hours: A Game Theory View

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Abstract

In this paper, we formalize a traffic game to illustrate traffic problems happened on buses and subway systems. We observe that in reality a main feature of these systems under peak hours is that severe congestion occurs only in one direction. We thus study an interesting detouring behavior that an individual may travel in the reverse direction for several stops to ensure getting on bus or on train. To analyze this behavior, another extensive form game is proposed, with its subgame modeled as an M/D/1 queue. We show existence of Nash equilibrium for the first traffic game with some additional restriction; and find ϵ-mixed strategy Nash equilibrium for both games by simulation. The experimental results indicates that rational individuals would detour when the traffic is congested enough; and that this strategy would possibly harm the traffic system.

Background

People in today's modern cities have been accustomed to the scene that thousands of people travels from uptown and suburban areas to downtown and urban areas every morning of workdays. This phenomena puts great stress on the traffic system, causing congestion at a specific period of a day, which is usually referred to as the morning peak. During morning peaks, bus stops and subway stations are filled with people who get up late and are hurrying up not to be late for work. Therefore, this competition for limited traffic resources among these workers naturally forms a game.

In this paper, a traffic game is formalize, abstracting the main features from this battle of peak hours. The ultimate goal of each player is to set off for work as late as possible while arriving before a deadline. To reflect the common rules of buses and subway systems, the traffic system adopts a first-come-first-serve(FCFS) rule with a fixed serving rate. Despite the inherent incontinuity of the ordering function exploited by FCFS rule, we show the existence of Nash equilibrium by modifying the original game with various approaches such as discretization or smoothing.

Aside from normal actions of queuing, a somewhat devious action, which we call detouring, is also taken into account. When Alice reaches a subway station and the queue is already very long, she may first travel in the reverse direction for several stops and then travels back, jumping the queue indirectly. Detouring may benefit some individuals, but it is a waste of the traffic capacity since the person travels longer. With more and more people adopting this strategy, social welfare diminishes. It is thus an example of the so-called 'involution' that the pressure of competition leads to bad results on every individuals. In this paper, we analyze the behavior of detouring as a subgame with rules of M/D/1 queue model, incorporating corresponding conclusions from Queuing Theory.

While a Nash equilibrium is hard to find in general, we simulate these two games and successfully find ϵ Nash equilibrium in an iterative manner.

Traffic game with no detouring

To simplify the game model and emphasize the main features of the traffic issue, we assume everyone starts from a same location for a same terminal points. The buses or trains run periodically with a constant frequency f. Each bus or train can only accommodate one passenger.

We define the traffic game as a 5-tuple Bayesian game $G = (\mathcal{I}, \mathcal{A}, \Theta, c, p)$.

- $\mathcal{I}$: a finite set of players.
- $\mathcal{A} = \{\mathcal{A}_i\}_{i=1}^n$: a set of actions for each player i.
- $\Theta = \{\Theta_i\}$: a set of types for each player i.
- $c = (c_1, \dots, c_n)$, where $c_i : \mathcal{A} \times \Theta_i \to \mathbb{R}$ is a cost function for player i. The utility function of player i is simply $u_i(a, \theta_i) = -c_i(a, \theta_i)$.
- p: a joint probability distribution $p(\theta_1, \dots, \theta_n)$ over types.

Further specifications are made in our discussion. Action spaces are homogeneous for each player $\mathcal{A}_i = [0, 1]$, corresponding to the time when each worker decides to set off for work. The spaces of type are also the same $\Theta_i = [0, 1]$, where the type of each player θ_i is drawn identically and independently from a distribution $p(\theta)$.

With the purpose of capturing the realistic scene, the cost function c is defined as: ($t \in \mathcal{A}$, $\theta_i \in \Theta_i$)

$$c_i(t, \theta_i) = \begin{cases} g(\theta_i - t_i), & \text{if player } i \text{ reaches in time,} \\ M, & \text{Otherwise.} \end{cases} \quad (1)$$

where $g(\cdot)$ captures the cost of getting up for work early, and M is the penalty of being late for work. More specifically, we assume the traffic system applies a first-come-first-serve rule with a constant frequency f, so that a player i reaches in time if and only if:

$$t_i + \max_{j=1,2,\dots,\sigma(i)-1} \left\{ \sigma(i) - j - \frac{t_i - t_{\sigma^{-1}(j)}}{f}, 0 \right\} \le \theta_i \quad (2)$$

where $\sigma : [0,1]^n \to S_n$ is the ordering function of $\{t_i\}_{i=1}^n$.

Existence of Nash equilibrium

A straightforward comment on the cost function is that it is not even continuous. As a result, it does not necessarily has a Nash equilibrium. Three possible independent approaches may be applied so as to ensure a Nash equilibrium.

- Modification 1: Discretize time $t \in [0,1]$ to $\epsilon\lfloor t/\epsilon \rfloor$, making G a finite game.
- Modification 2: For any different players i, j with $t_i = t_j$, they can be transported simultaneously. In other words, they have different ordering function σ, which prioritize themselves over other players with the same action, respectively.
- Modification 3: Smooth the queuing function σ. t_i is ranked before t_j iff $t'_i < t'_j$ where t'_i is drawn from a probabilistic distribution $\ell(t_i)$. The ambiguity of $t'_i = t'_j$ is removed, since it happens with zero probability.

Followed directly from Nash Theorem (1951), the traffic game G_1 with modification 1 has a mixed strategy Nash equilibrium. However, a crucial drawback of applying modification 1 is that it makes strategy space too large for computation. It is also a trivial inference that with modification 3, the game G_3 possesses a pure strategy Nash equilibrium (using Debreu, Glicksberg, Fan, 1952). Moreover, with modification 2, a pure strategy Nash equilibrium can be surprisingly proved to exist in G_2 as well, with some additional restriction.

Theorem 1. *With modification 2, there exists a pure strategy Nash equilibrium for traffic game $G = (\mathcal{I}, \mathcal{A}, \Theta, c, p)$ if:*

- *$\{\theta_i\}_{i=1}^n$ is fixed and public;*
- *g is continuous, monotone increasing and bounded on domain $[0,1]$;*
- *The penalty $M \ge g(1)$.*

Proof. It has been shown (Dasgupta and Maskin 1986) that a game G with actions and corresponding utility (A_i, u_i) possesses a pure strategy Nash equilibrium, if u_i is upper semi-continuous and graph-continuous, and quasi-concave in a_i; while A_i is nonempty, convex and compact.

For simplicity, we only prove the most difficult statement that $u_i(t, \theta_i)$ is upper semi-continuous in t with θ_i fixed. To see this, we shall first claim that the feasible space $\mathcal{A}^*$ in which i is not late for work is compact. (The boundary condition for θ_i is either condition 2 or $\theta_i \ge \theta_j$ for some $j \ne i$, ensuring that it is closed.) Remember that $u_i(t, \theta_i) = -g(\theta_i - t_i)$ for $t \in \mathcal{A}^*$ and g is continuous, any sequence $\{t^n\}$ with $\lim_{n\to\infty} t^n \to \bar{t}$ and $t \in \mathcal{A}^*$ would have $\lim_{n\to\infty} u_i(t^n, \theta_i) = u_i(\bar{t}, \theta_i)$.

The only possible sequence that breaks the continuity is $\{t^n\}$ with $t^n \notin \mathcal{A}^*$ but $\bar{t} = \lim_{n\to\infty} t^n \in A^*$. However, $t^n \in \arg\min_{t'} u_i(t', \theta_{-i})$, given $M \ge g(1) \ge g(x), \forall x \in [0,1]$. Thus, it must be the case that

$$\lim_{n\to\infty} u_i(t^n, \theta_i) \le u_i(\bar{t}, \theta_i) \quad (3)$$

□

Despite the existence of Nash equilibrium in G when θ is fixed and public, the game remains complicated if θ is secret as in reality. With curiosity of finding a Bayesian Nash equilibrium, it would be great to find a strategy profile σ, such that

$$\sigma_i(\theta_i) \in \arg\max_{\sigma'_i} U_i(\sigma'_i, \sigma_{-i}, \theta_i) \quad (4)$$

where $U_i(\sigma'_i, \sigma_{-i}, \theta_i)$ is the expected payoff of player i of type θ_i:

$$\begin{aligned} U_i(\sigma'_i, \sigma_{-i}, \theta_i) &= \sum_{\theta_{-i}} p(\theta_{-i} \mid \theta_i) u_i(\sigma'_i, \sigma_{-i}(\theta_{-i}), \theta_i, \theta_{-i}) \\ &= \sum_{\theta_{-i}} \prod_{j \ne i} p(\theta_j) u_i(\sigma_i, \sigma_{-i}(\theta_{-i}), \theta_i) \quad (5) \end{aligned}$$

However, the complexity of condition 2 makes it not realistic for finding such a Nash equilibrium. Therefore, we would simulate the traffic game to find an ϵ-approximate Nash equilibrium instead. We shall emphasize that this is not too strong an assumption in reality. As long as $M \ge g(1) + \epsilon$, players will not be satisfied by being late if setting off earlier helps.

Traffic game with detouring

It has been pointed out that a strategy of detouring may benefit an individual at least in a short term, when a player observes a long queue at a subway station. In this section, we formalize an extensive form game G_d with imperfect information (d stands for detouring) that takes this devious strategy into account. For simplicity, the arrival deadline θ_i of each player is assumed to be public.

$G_d = (\mathcal{I}, \mathcal{A}, H, \mathcal{J}, P, \{u_i\})$. $(\mathcal{I}, \{u_i\}$ defined the same as G).

- $\mathcal{A} = \{\mathcal{A}_i\}_{i=1}^n$.
- H: the set of history.
- $\mathcal{J}$: a partition of non-terminal histories.
- $\mathcal{P}$: an assignment of $J \in \mathcal{J}$ to $\mathcal{I}$.

Each player $i \in \mathcal{I}$ makes two decisions in the whole game tree. For the first decision, $A_i(J_{start,i}) = [0,1]$, in which i picks a favored time to set off. For the second decision, player i chooses an action from $A_i(J_k) = \{\text{Join}, \text{Detour by } x\}$, where k is the queuing length when A_i reaches the subway station.

For clarity, think of n players choosing t_i simultaneously. We adopt the modification 3 in the previous section, so that i reaches the subway station at $t'_i \sim \ell(t_i)$. They then play an extensive form game G'_d in the order of $\sigma(t')$, choosing from

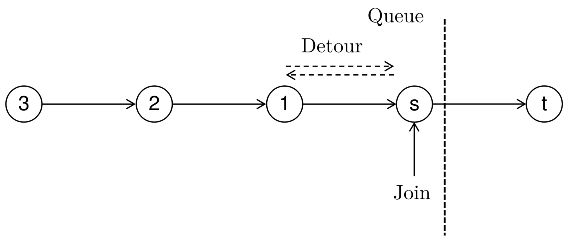

Figure 1: Traffic model with detouring behavior

$\{$Join, Detour by $x\}$ with observation of the queue length at t'_i. If some players i choose to Detour by x, they travel back for x stations to get on the bus.

For a better understanding of the detouring behavior, let us only consider the subgame G'_d, where each player is given with t'_i and utility function $u_i(a_i) = -t_{i,\text{arrive}} + t'_i$.

Theorem 2. *A pure strategy $s_i \in \mathcal{A}_i$ is a never best response, if at an information set J_k of G'_d with $P(J_k) = i$ and $k \geq 3$, $s_i(J_k) =$ Join.*

Proof. Suppose the queue has players $\{q_1, q_2, \dots, q_k\}$. With strategy s_i, q_3 successfully takes on the m-th train. Let $s'_i =$ Detour by $\lfloor \frac{m-1}{2} \rfloor$ ($m \geq 3$ so $\lfloor \frac{m-1}{2} \rfloor \geq 1$). Then with s'_i, i must be able to get on the m-th train. This is because all j with $t'_j < t'_i$ does not take this train before q_3 does; while all j with $t'_j > t'_i$ meets the m-th train later than i does. $\square$

Corollary 3. *Detouring happens as long as there are four rational players i, j, k, l with $t'_i < t'_j < t'_k < t'_l \leq t'_i + 1/f$.*

Several more interesting analyses can be deduced with the help of queuing theory. In this paper, we only discuss a simple case of $M/D/1$ model, in which the arrival pattern t' subjects to a Markovian process and the serving rate is deterministic. We shall especially point out that the length of queue includes everyone in the traffic system.

Theorem 4. *An M/D/1 queue with arrival rate λ and serving rate μ has a stationary state when $\rho = \frac{\lambda}{\mu} < 1$ with:*

- *Average length of queue $L_s = \frac{\lambda}{\mu} + \frac{\lambda^2}{2\mu(\mu-\lambda)}$.*
- *The probability distribution of length is a bit more complicated, but its generating function can be given by: $P(x) = \sum_{n=0}^{\infty} p_n x^n = \frac{(1-\rho)(1-x)}{1-xe^{\rho(1-x)}}$. (See (Jain and Sigman 1996))*

An straightforward observation is that choosing strategy $s_i =$ Detour by x actually helps a player can get on train if the queue length keeps $L \leq x$ for $f \cdot x$ time. For $s_i =$ Join, this happens immediately when $L = 0$. However, this is not a necessary condition for getting on train. The actual condition of getting on train is too complicated to depict analytically.

Simulation

Experiment 1

The simulation starts with the traffic game with no detouring behavior. In our first experiment, N players are with θ_i chosen from Gaussian distribution $\mathcal{N}(\mu, \sigma)$ (clipped to $[0, 1]$) and fixed. With random initialization, the strategy profile is explored greedily by each player and updated with discounted step size. The exploration stops as soon as the players find an ϵ-Nash equilibrium.

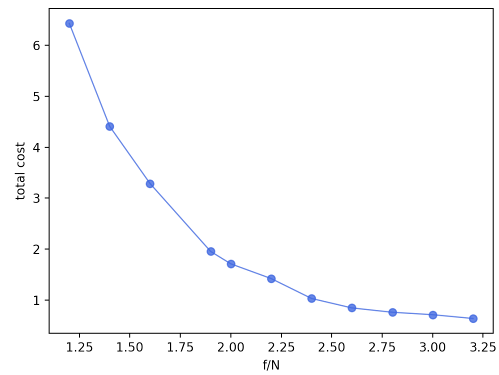

Figure 2: The total cost of the found ϵ-Nash equilibrium ($\epsilon = 0.05, N = 10, \mu = 0.6, \sigma = 0.2$)

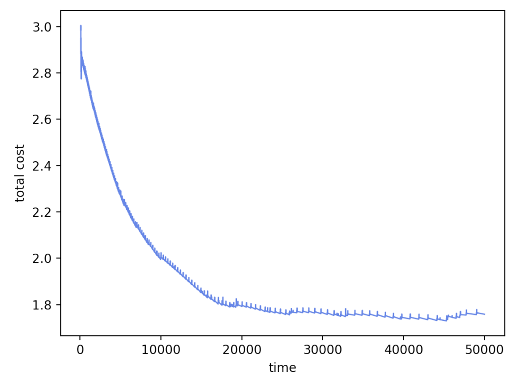

Figure 3: The convergence of total cost ($N = 10, f = 20$)

Figure 2 exhibits the total cost of ϵ-Nash equilibrium found by our iterative method. The cost increases drastically when f/N under 2. This phenomena is caused by the concentration of θ_i among a small period of time. It is discovered that the greedily updating algorithm do stably converges to an equilibrium (see figure 3).

Experiment 2: Detouring

The purpose of experiment 2 is to find an approximate Nash equilibrium for the subgame G'_d as described in previous sections. Suppose there are K stations in total, forming a huge $M/D/1$ queue with $\rho = \frac{\lambda}{f}$.

We formalize G'_d with a state space $Q \subseteq \mathbb{N}^K \times 2^K$, denoting the number people waiting at each station and whether the K trains are empty. A strategy s of a player as an $4 \times K$ matrix, where $\sum_{j=1}^{k} s_{ij} = 1$ for all i. According to our pre-

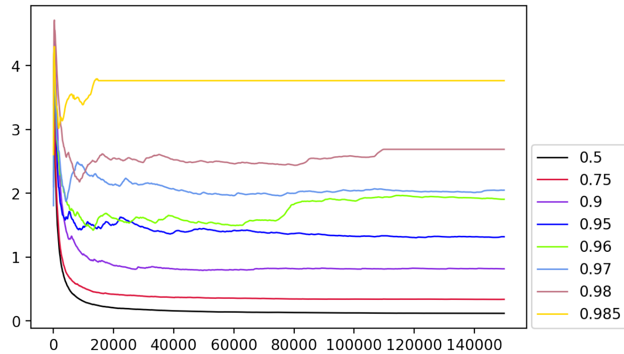

Figure 4: The convergence of multiplicative weight algorithm on different setting of ρ.

vious analysis (Theorem 2), a player cannot choose to join the queue when length ≥ 3, so $s_{4,1} = 0$. Consider a homogeneous equilibrium that everyone players the same strategy s. Each strategy s must correspond to a stable probability distribution on Q when $\rho < 1$ (Theorem 4), denoted by $p(q \mid s)$,$\sum_{q \in Q} p(q \mid s) = 1$. A Nash equilibrium of G'_d requires that $p(q \mid s)$ is consistent with s, in the sense that s must be a best response of $p(q \mid s)$.

To reach such a Nash equilibrium (approximately), we introduce multiplicative weight algorithm (Littlestone and Warmuth 1994). We estimate the cost $c(s_t)$ of strategy s_t at its stable point $p(q \mid s_t)$ by taking an average on the first T arriving passengers. And then the multiplicative weight algorithm is applied to update s_t:

$$w_{ij}^{t+1} = w_{ij}^{t} \cdot (1-\epsilon)^{c(s_t)}, 1 \leq i \leq 4, 1 \leq j \leq K. \quad (6)$$

where s_t chooses action j when observing i with probability w_{ij}^{t}/W_t. Experiments show that multiplicative weight algorithm indeed converge to an ϵ-Nash equilibrium. We simulate this game by 500,000 seconds with $f = 20$ and $K = 5$. The arrival rate λ varies from 10 to 19.8, which yields the results shown by figure 4.

With the same settings, it is also discovered that the uncertainty of average cost when reaching ϵ-Nash equilibrium increases rapidly when $\rho > 0.9$. (See figure 6. The black line stands for the social optimal choice, as $t = \left(\rho + \frac{\rho^2}{2(1-\rho)}\right)/f$ indicated by theorem 4.) The social welfare deviates from the optimal choice, as a result of the increasing probability of detouring action. Figure 5 compares the mixed strategy at ϵ-Nash equilibrium when the game G'_d is set with different ρ. In fact, when $\rho < 0.9$, the probability of choosing 'Join' dominates every other detouring actions, except the cases that the queuing length is greater or equal 3. Once $\rho > 0.9$, however, passengers would prefer detouring for its own utility, which may exacerbate the traffic jam and reduce social welfare in the end.

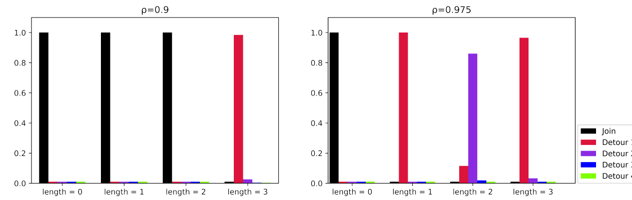

Figure 5: The strategy profile at ϵ-Nash equilibrium when for different ρ

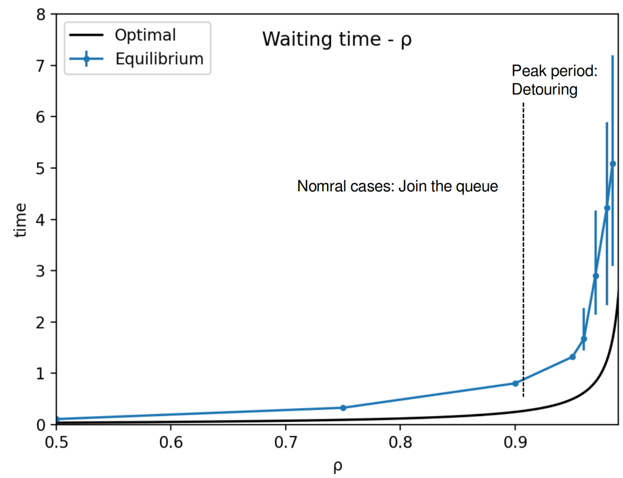

Figure 6: Comparing the optimal expected waiting time (black) and the average waiting time of the found equilibrium (blue).

Conclusion

In this paper, we defined, analyzed and simulated two games based on real-world observations of common traffic systems. The first game is modeled as a basis of the second. We proposed three possible modifications to the incontinuous traffic game and proved the existence of Nash equilibrium on the modified games. To study the detouring behavior, we modeled the second game as a stochastic process and borrowed corresponding conclusions from queuing theory. After analysis and simulation, we found out a tendency of individuals to choose the detouring behavior during peak hours, which may harm the traffic system as a whole.

Appendix

All python codes used for simulating traffic games G and G'_d could be found at https://github.com/Qianhewu/Traffic-Game.

Some additional definitions and explanations are listed below for integrity.

Definition 1. *A function $u_i : \mathcal{A} \to \mathbb{R}$ is upper semi-continuous if for any sequence $\{a^n\} \subseteq \mathcal{A}$ such that $\lim_{n\to\infty} a^n \to \overline{a}$,*

$$\lim_{n\to\infty} u_i(a^n) \leq u_i(\overline{a}) \quad (7)$$

Definition 2. *A function $u_i : \mathcal{A} \to \mathbb{R}$ is graph continuous if for any $\overline{a} \in \mathcal{A}$ there exists a function $F_i : \mathcal{A}_{-i} \to \mathcal{A}_i$*

with $F(\overline{a}_{-i}) = \overline{a}_i$*, such that* $u_i(F_i(a_{-i}), a_i)$ *is continuous at* $a_{-i} = \overline{a}_{-i}$*.*

The concept of ϵ-Nash equilibrium is used excessively throughout our simulation. The exact definition is as follows.

Definition 3. *A strategy profile* s *is an* ϵ*-Nash equilibrium for a normal form game* $G = (\mathcal{A}, \{u_i\})$ *if for all* i*:*

$$u_i(s_i, s_{-i}) \geq u_i(s'_i, s_{-i}) - \epsilon, \forall s'_i \in \mathcal{A}_i. \tag{8}$$

We specify the greedy update applied in experiment 1 as

$$t_i^{(k+1)} = \frac{\theta_i - \tau_i^{(k)}}{k}\mathbb{I}[\theta_i - \tau_i^{(k)} \geq \alpha]+$$

$$\frac{\theta_i - \tau_i^{(k)} - \epsilon}{k}\mathbb{I}[\theta_i - \tau_i^{(k)} < \alpha] + (\theta_i - \tau_i^{(k)})\mathbb{I}[\theta_i - \tau_i(k) < 0] \tag{9}$$

where $\tau_i^{(k)} = t_i + \max_{1 \leq j \leq \sigma(i)-1}\left\{\sigma(i) - j - \frac{t_i - t_{\sigma^{-1}(j)}}{f}\right\}$ is the arrival time of player i.

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