Topology

Wenda

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1 Topological Space

Continuous mappings $f : \mathbb{R}^n \to \mathbb{R}^m$ is usually defined using metrics: $\forall \epsilon > 0, \exists \delta > 0$, s.t. $d(x, x_0) < \delta$ gives $d(f(x), f(x_0)) < \epsilon$. But can we define continuity without any notion of metric?

The answer is yes. We can define continuity using only topological structures on sets.

1.1 Basic Definition

Definition 1.1.1 (Topological Space)**.** *Suppose we have a set X. A topology $\mathcal{T}$ is a set of the subsets of X with the following properties:*

1. *$\varnothing, X \in \mathcal{T}$.*
2. *(Closed under arbitrary union) $\forall i \in \mathcal{I}$, $U_i \in \mathcal{T}$, then $\cup_{i \in \mathcal{I}} U_i \in \mathcal{T}$.*
3. *(Closed under finite intersection) $U_1, \ldots, U_n$, then $\cap_{i=1}^n U_i \in \mathcal{T}$.*

Remark. A subset $U \in \mathcal{T}$ is also called an open set. A subset $U \in X$ is called a closed set if and only if $X - U$ is open.

Some examples.

- $\mathcal{T} = \{\varnothing, X\}$, trivial topology.
- Power set: $\mathcal{T} = P(X) := \{\text{All subsets of } X\}$, dubbed discrete topology.

Definition 1.1.2 (Metric)**.** *Let X be a set. A metric on X is a mapping $X \times X \to \mathbb{R}_{\geq 0}$, subject to the following.*

1. *$d(x, y) = 0 \Leftrightarrow x = y$.*
2. *$d(x, y) = d(y, x), \forall x, y \in X$.*
3. *(Triangular inequality) $d(x, y) + d(y, z) \geq d(x, z), \forall x, y, z \in X$.*

Definition 1.1.3 (Open Ball)**.** *Let X be a set and d is a metric on X. We define an open ball at x_0 with radius ϵ as*

$$B(x_0, \epsilon) := \{x \in X \mid d(x, x_0) < \epsilon\}. \tag{1}$$

Definition 1.1.4 (Metric Topology)**.** *Let X be a set and d is a metric on X. Then for any subset $U \in X$, U is open if and only if*

$$\forall x \in U, \exists \epsilon > 0, s.t. B(x, \epsilon) \subseteq U \tag{2}$$

Remark. This definition satisfies all the requirements of topological space. The proof is trivial.

Moreover, all subset $U \in \mathcal{T}$ can be written as the union of open balls.

$$U = \cup_{x \in U} B(x, \epsilon). \tag{3}$$

Definition 1.1.5 (Subspace Topology)**.** *Let $(X, \mathcal{T})$ be a topological space. Suppose $S \subset X$ is a subset of X, then a subspace topology on S induced by $\mathcal{T}$ is*

$$\mathcal{T}' = \{U \in S \mid U = S \cap V, V \in \mathcal{T}\}. \tag{4}$$

Remark. Note that an open set $U \subset S, U \in \mathcal{T}'$ is not necessarily open at $(X, \mathcal{T})$ (i.e., $U \in \mathcal{T}$). A simple example is $S = [0, 1] \subset \mathbb{R}$ and $\mathcal{T} = \{(a, b) \mid a, b \in \mathbb{R}\}$, where $U = [0, 0.5) \in \mathcal{T}'$ but $U \notin \mathcal{T}$.

1.2 Continuous Mapping

Now we have defined topology on sets. It's time to try defining continuity without using the notion of metric. Remember that in metric topology, we can define continuity as $\forall \epsilon > 0, \exists \delta > 0$, $|x - x_0| < \delta$ yields $|f(x) - f(x_0)| < \epsilon$. This is in fact equivalent to $f^{-1}\big(f(x_0) - \epsilon, f(x_0 + \epsilon)\big)$ is open in $\mathbb{R}$.

Definition 1.2.1 (Continuous Mapping)**.** *Let X, Y be two topological space. We call a mapping $f : X \to Y$ continuous, if*

$$\forall U \subseteq Y \text{ that is open}, f^{-1}(U) \subseteq X \text{ is open}. \tag{5}$$

Remark. The composition of continuous mappings is also continuous. Also observe that definition 1.2.1 reduces to the continuous mapping in Calculus when Y is a metric topology, in which case we can show $\underline{\forall x \in X, \epsilon > 0, \exists U \in \mathcal{T}_X \text{ with } x \in \mathcal{T}_X, U \subseteq B(f(x), \epsilon)}$ is equivalent to definition 1.2.1.

Definition 1.2.1 also provides us a useful trick for defining a topology in space X. Let $\mathcal{T}_X^f := \{f^{-1}(U) : U \in \mathcal{T}_Y\}$, then $\mathcal{T}_X^{(f)}$ is a topology on $f^{-1}(Y) \subseteq X$.

The definition of continuous mapping bridges across two topological space.

Definition 1.2.2 (Homeomorphism)**.** *Let X, Y be two topological space. A mapping $f : X \to Y$ is a homeomorphism, denoted $X \cong Y$, if*

1. *f is continuous*
2. *f is bijective.*
3. *f^{-1} is also continuous.*

Remark. Homeomorphism is typically hard to disprove. It would take some effort to proof $\mathbb{R} \ncong \mathbb{R}^2$, and even more effort to show $\mathbb{R}^2 \ncong \mathbb{R}^3$.

Some examples.

- (Embedding) Let $f : X \to Y$ be continuous and injective. f is called an embedding from X to Y, if $f : X \to f(X)$ is homeomorphism.
  $f : \mathbb{R} \to \mathbb{R}^2$, $f(x) = (ax, bx)$ is an embedding.
- (Stereographic projection) $S^2 = \{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 + z^2 = 1\}$. Then $S^2 \backslash \{(0, 0, 1)\} \cong \mathbb{R}^2$.
  $\varphi(x, y, z) = (\frac{2x}{1-z}, \frac{2y}{1-z} - 1)$ and $\varphi^{-1}(x, y-1) = (\frac{4x}{4+x^2+y^2}, \frac{4y}{4+x^2+y^2}, \frac{-4+x^2+y^2}{4+x^2+y^2})$

Next, we build a useful tool for proving continuous mappings. The following lemma allows us to check the continuity of a mapping by checking continuity in its segments.

Lemma 1.2.3 (Pasting Lemma). *Let X, Y be two topological spaces, $\{X_i \mid i \in I\}$ be a family of subspaces of X. Let $\{f_i \mid i \in I\}$ be a family of continuous functions and $f_i|_{X_i \cap X_j} = f_j|_{X_i \cap X_j}$ for $\forall i, j \in I$. If*

- *X_i are open sets in X.*
- *Or, X_i are closed and I is finite.*

Then $f : X \to Y$ with $f|_{X_i} = f_i$ is continuous.

1.3 Products of Topology

Let X and Y be two topological space. We want to find topology on $X \times Y$. However, $U \times V, U \in \mathcal{T}_X, V \in \mathcal{T}_Y$ does not naturally form a topology, as it is not closed under union.

Definition 1.3.1 (Product Topology). *Let $(X, \mathcal{T}_X)$, $(Y, \mathcal{T}_Y)$ be two topological space. We define open boxes in $X \times Y$ as $U \times V$, where $U \in \mathcal{T}_X$ and $V \in \mathcal{T}_Y$.*

Moreover, we call a subset $W \subseteq X \times Y$ open, if it is a union of open boxes.

Remark. To check this is indeed a topology is trivial. The most crucial property is that intersecting two open boxes $(U_1 \times V_1) \cap (U_2 \times V_2) = (U_1 \cap U_2) \times (V_1 \cap V_2)$ is still an open box.

1.4 Basis

We have seen in previous examples that a topology space may be constructed from some basic ingredients, e.g., open balls or open boxes. In this section, we formalize this intuition by characterising open balls and open boxed as basis.

Definition 1.4.1 (Basis for a topology). *Let X be a set. $\mathcal{B}$ be a family of subsets of X. We call $\mathcal{B}$ a basis for a topology if*

- *(Covers X) $\forall x \in \mathcal{B}, \exists U \in \mathcal{B}$, such that $x \in U$.*
- *(Intersection) If $U, V \in \mathcal{B}$, there exists a collection of subsets $\{W_i \in \mathcal{B} \mid i \in I\}$ such that $U \cap V = \cup_{i \in I} W_i$.*

Definition 1.4.2 (Subbasis). *Let X be a set. $S \subseteq 2^X$ is a subbasis if*

$$\bigcup_{S_i \in S} S_i = X. \quad \textit{(only need S cover X)} \tag{6}$$

Given a subbasis, how do we generate a basis?

$$\text{Subbasis } S \longrightarrow \text{Basis } \mathcal{B} \longrightarrow \text{Topology } \mathcal{T} \tag{7}$$

Let $\mathcal{B}_S = \{S_{i_1} \cap \cdots \cap S_{i_m} \mid S_{i_1}, \ldots, S_{i_m} \in S\}$. Then $\mathcal{B}_S$ is a basis.

Let $\mathcal{T}_\mathcal{B} = \{\bigcup_{i \in \mathcal{I}} B_i \mid B_i \in \mathcal{B}\}$. Then $\mathcal{T}_\mathcal{B}$ is a topology.

Examples.

- Metric topology (X, d), corresponding to basis open balls $\{B(x, \epsilon) \mid x \in X, \epsilon > 0\}$.
- Product topology $X \times Y$, corresponding to basis open boxes $\{U \times V \mid U \in X, V \in Y\}$.
- $S = \{(-\infty, a) \mid a \in \mathbb{R}\} \cup \{(b, \infty) \mid b \in \mathbb{R}\}$ is a subbasis in $\mathbb{R}$. Its corresponding basis is $\mathcal{B}_S = \{(a, b) \mid \forall a, b \in \mathbb{R}\}$ which is already a topology in $\mathbb{R}$.
- (**Initial topology**) Consider some continuous mappings $f_i : X \to Y_i$. Let $S = \{f_i^{-1}(U) \mid U \subset Y_i \text{ open}\}$. Then $\mathcal{T}_S$ is called an initial topology.

Proposition 1.4.3. *$f : X \to Y$, S is a subbasis of the topology of Y. Then*

$$f \text{ is continuous} \Leftrightarrow f^{-1}(S_i) \in \mathcal{T}_X, \forall S_i \in S \tag{8}$$

1.5 Hausdorff Space

Definition 1.5.1 (Hausdorff space)**.** *A topological space X is called Hausdorff if for any pair of points $x, y \in X$, there exists an open neighborhood U of x and an open neighborhood V of y such that $U \cap V = \varnothing$.*

2 Quotient Topology and Gluing

2.1 Basic Definition

Relation. A relation in X is a subset $R \subset X \times X$. $(x, y) \in R$ is denoted $x \sim y$.

An **equivalent relation** satisfies the following properties:

- (reflexive) $x \sim x$
- (symmetric) $x \sim y \Rightarrow y \sim x$
- (transitive) $x \sim y, y \sim z \Rightarrow x \sim z$

Consider two mappings $p_1, p_2 : T \to X$. We can define $p : T \to X \times X$, such that $p(t) = (p_1(t), p_2(t))$. Then $R_{p_1 p_2} = p(T)$ is a relation generated by p_1 and p_2.

Definition 2.1.1 (Equivalent class)**.** *Let $\sim$ be an equivalent relation. An equivalent class containing x is denoted by*

$$[x] := \{y \in X \mid x \sim y\}. \tag{9}$$

Definition 2.1.2 (Quotient sets)**.** *The quotient set in X of an equivalent relation $\sim$ is denoted by*

$$X/\sim := \{[x] \mid x \in X\}. \tag{10}$$

2.2 Quotient Topology

Definition 2.2.1 (Quotient Topology)**.** *$(X, \mathcal{T}_X)$ is a topology space. Let $f : X \to Y$ be a surjective quotient. $U \subset Y$ is open if $f^{-1}(U)$ is open. The quotient topology is defined as*

$$\mathcal{T}_Y = \{U \subset Y \mid f^{-1}(U) \in \mathcal{T}_X\} \tag{11}$$

Remark.

- $\mathcal{T}_Y$ is the finest topology that makes f continuous.
- For an equivalent relation $\sim$, consider its projection map $\pi : X \to X/\sim$. The quotient topology is
$$\mathcal{T} = \{U \subset X/\sim \ \mid \pi^{-1}(U) \in \mathcal{T}_X\} \tag{12}$$

Definition 2.2.2 (Quotient map)**.** *Let X and Y be topological spaces. Suppose $f : X \to Y$ is a continuous and surjective. If the quotient topology is homeomorphic to the topology on Y, i.e., $(X/\sim_f, QT) \cong (Y, \mathcal{T}_Y)$, then f is called a quotient map.*

The question is: *how do we decide a map is a quotient map or not?* The following propositions 2.2.3 and 2.2.4 are usually used jointly to prove a quotient map.

Proposition 2.2.3. *Let $f : X \to Y$ be continuous and surjective. Let $\{U_i\}_{i \in \mathcal{I}}$ be an open cover of Y. We denote the restriction of f on U_i by $f_i : f^{-1}(U_i) \to U_i$ with $f_i = f|_{f^{-1}(U_i)}$.*

Proposition 2.2.4. *Let $f : X \to Y$ be continuous and surjective. If f has a section $S : Y \to X$, i.e., $S : Y \to X$ continuous and $f \circ S = Id_Y$, then f is a quotient map.*

Remark. The existence of a section S implies f is surjective.

Proposition 2.2.5. *Let $f : X \to Y$ be a continuous, surjective map and f is open (or closed) map. Then f is a quotient map.*

2.3 Gluing Open Subsets

Definition 2.3.1 (Open gluing datum)**.** *An open gluing datum consists of the following:*

- *(What to glue) A family of topological spaces $\{U_i\}_{i \in \mathcal{I}}$.*
- *(Where to glue) Open subspaces $U_{ij} \subset U_i$, we also set $U_{ii} = U_i$.*
- *(How to glue) homeomorphism $f_{ij} : U_{ij} \to U_{ji}$*

satisfying the following

- $f_{ii} = Id_{U_i}$
- $f_{ij}(U_{ij} \cap U_{ik}) \subset U_{ji} \cap U_{jk}$
- *(Cocycle condition) The homeomorphisms agree on triple intersections*

$$f_{ik}|_{U_{ij} \cap U_{ik}} = f_{jk}|_{U_{ji} \cap U_{jk}} \circ f_{ij}|_{U_{ij} \cap U_{ik}} \tag{13}$$

Remark. The cocycle condition guarantees the consistency when gluing > 2 subsets together.

Definition 2.3.2 (Disjoint union)**.** *The disjoint union of a collection of sets $\{U_i\}_{i\in\mathcal{I}}$:*

$$\bigsqcup_{i\in\mathcal{I}} U_i = \{(x,i) \mid i \in \mathcal{I}, x \in U_i\} \tag{14}$$

Definition 2.3.3 (Open gluing: (U_i, U_{ij}, f_{ij}))**.**

$$U = \bigsqcup_{i\in\mathcal{I}} U_i. \tag{15}$$

$$\sim: (x,i) \sim (y,i) \Leftrightarrow x \in U_{ij}, y \in U_{ji}, f_{ij}(x) = y. \tag{16}$$

Check that this is indeed an equivalence relation by definition. We call $(U/\sim, QT)$ the topological space glued from datum (U_i, U_{ij}, f_{ij}).

2.4 Manifold

Definition 2.4.1 (Chart)**.** *Let X be a topological space. An n-dimensional chart on X consists of an open subset $U \subset X$ equipped with the subspace topology and a homeomorphism $\phi U \to V$, where V is an open subspace of $\mathbb{R}^n$ with the standard topology.*

Definition 2.4.2. *Let M be a Hausdorff space with a countable topological basis (also called second countable). M is a topological manifold (of pure dimension n) if there exists a collection of (n- dimensional) charts $(U_i, \varphi_i)_{i\in I}$ such that $M = \bigcup_{i\in I} U_i$.*

3 Connectedness

3.1 Connected Spaces

Definition 3.1.1 (Connected Space)**.** *A topological space X is connected if it is not the union of two disjoint non-empty open subsets.*

Remark.

X is connected. $\Leftrightarrow$ The subsets that are open and closed in X must be $\varnothing$ and X.

3.1.1 Connectedness and continuous maps

Proposition 3.1.2. *Let $f : X \to Y$ be a continuous map of topological spaces. Assume X is connected, then $f(X)$ is connected.*

In order words, connectedness is a *topological invariant*.

Using this property, we can easily show $\mathbb{R} \ncong \mathbb{R}^2$. Since $f(\mathbb{R}\backslash\{0\})$ is a connected subspace of $\mathbb{R}^2$.

3.1.2 Connectedness and subspaces

Proposition 3.1.3. *Let X be a topological space and $X' \subseteq X$ is a connected dense subset, then X is also connected.*

Remark. A subset $A \subseteq X$ is **dense**, if its closed envelop $\overline{A} = X$, where $\overline{A} = \bigcap_{i \in I} V_i$, V_i is closed and $A \subseteq V_i$. An example: $\overline{\mathbb{Q}} = \mathbb{R}$.

Proof. Suppose X is not connected, then there exists a nonempty proper subset $U \subset X$ that is closed and open. Since X' is dense, $U \cap X' \neq \varnothing$. Then $U \cap X' \subseteq X'$ is closed and open in X', which means $U \cap X' = X'$. Thus, $U = X' = \overline{X}' = X$, contradict. $\square$

Corollary 3.1.1. If $A \subseteq X$ and A is connected, then its closed envelop $\overline{A}$ is also connected

3.1.3 Locally constant function

Definition 3.1.4 (Locally constant function)**.** *Let Λ be a topological space with discrete topology. A continuous map $f : X \to \Lambda$ is locally constant if*

$$\forall x, \exists U \in \mathcal{T}, x \in U, \text{ such that } f|_U \text{ is constant.} \tag{17}$$

Proposition 3.1.5. *Let $(\Lambda, discrete)$ be a topological space with at least two elements. X is a topological space. Then X is connected if and only if any locally constant function $f : X \to \Lambda$ is constant.*

Using proposition 3.1.5, we can prove the following convenient proposition for connectedness.

Proposition 3.1.6. *Let $\{A_i\}_{i \in I}$ be a collection of connected subspaces of X. Let B be a connected subspace such that $B \cap A_i \neq \varnothing$, then*

$$B \cup \bigcup_{i \in I} A_i \text{ is connected.} \tag{18}$$

Proof. Define $A = B \cup \bigcup_{i \in I} A_i$. Let $f : A \to \Lambda$ be a locally constant function. We know $f|_{A_i}$ and f_B are all constant function as A_i and B are connected. Thus, $f|_A = f|_U$ is constant. $\square$

Corollary 3.1.2. $(X, \mathcal{T})$, if $\forall x, y \in X$, there exists a connected subspace U, $x, y \in U$, then X is connected.

Corollary 3.1.3. X, Y are connected $\Rightarrow X \times Y$ is connected.

Corollary 3.1.4. $f : X \to Y$ is a surjective continuous map. Y has the quotient topology. Then X is connected $\Rightarrow Y$ is connected.

Remark. Based on these corollaries above, we can show some basic spaces are connected.

- $\mathbb{R}^n$ is connected. (from corollary 3.1.3)
- S^n is connected. *Proof:* for $x, y \in S^n$, choose a $z \in S^n$ and $z \neq x, z \neq y$. Then $\mathbb{R}^n / \{z\} \cong \mathbb{R}^n$ is connected.
- $\mathbb{R}P^n$ and $\mathbb{C}P^n$ connected from corollary 3.1.4.

3.2 Path Connectedness

Definition 3.2.1 (Path). *Let X be a topological space. A path γ is a continuous map $\gamma : I \to X$. $\gamma(0)$ and $\gamma(1)$ are the start and the end of the path.*

Definition 3.2.2 (Path connected). *A topological space X is path connected if for any $x, y \in X$, $\exists$ a path γ with $\gamma(0) = x$, $\gamma(1) = y$.*

3.3 Connected Components

Definition 3.3.1 (connected components). *$(X, \mathcal{T})$. x, y are mutually connected if there is a connected subspace $U \subset X$ and $x, y \in U$.*

Mutual connectedness is an equivalence relation. Its equivalence class are the connected components of X.

Total number of connected components is a topological invariant: $\pi_0(X)$.

Without any difficulty we can extend the mutual connectedness to mutual path connectedness. Note that mutual path connectedness is also an equivalence relation. Similarly we define path connected components.

4 Compactness

Compactness seeks to generalize the property of a bounded and closed subset of Euclidean space.

Definition 4.0.1 (Compactness). *Let $X, \mathcal{T}$ be a topological space. C is compact if $\forall$ open cover $\{u_i\}_{i \in \mathcal{I}}$ of X, there exists $U_{i_1}, \ldots, U_{i_n} \in \{U_i\}_{i \in \mathcal{I}}$, such that $X \subset \bigcup_{i=1}^{n} U_{i_j}$.*

Remark. In other words, any open cover of X contains a finite open cover of X.

Examples.

- $(X, trivial)$ is compact. If $|X| < \infty$, $(X, discrete)$ is compact.
- $[0, 1]$ is compact.
- $(0, 1)$ is not compact, because $\{(\frac{1}{n}, 1)\}_{n=1}^{\infty}$ is an open cover, but no finite subset of it is a cover.

4.1 Basic Property of Compact Spaces

Proposition 4.1.1. *If f is continuous, X is compact, then $f(X)$ is also compact.*

4.1.1 With subspace

Proposition 4.1.2. *X is a compact, closed space. Z is a closed subspace of X. Then Z is also compact.*

Remark. This proposition 4.1.2 guarantees that every bounded closed subset of $\mathbb{R}$ is compact.

Proposition 4.1.3. *X is Hausdorff. $Z \subset X$ is compact $\Rightarrow$ Z is closed.*

4.1.2 With product space

Definition 4.1.4 (Refinement of an open cover). *$\mathcal{U} = \{U_i\}_{i\in I}, \mathcal{V} = \{V_i\}_{i\in I}$ are open covers of X. If $\forall V_i, \exists U_k$, such that $V_i \subset U_k$, then $\mathcal{V}$ is called a refinement of $\mathcal{U}$.*

Proposition 4.1.5. *If X, Y are compact topological spaces, then $X \times Y$ is also compact.*

Proof Sketch. Let $\mathcal{U}$ be an open cover of $X \times Y$, we can find a refinement of $\mathcal{V}$ of $\mathcal{U}$, where $\mathcal{V}$ only contains the open boxes in $X \times Y$. We can easily find a finite open cover in $\mathcal{V}$ using compactness of X and Y. $\square$

Theorem 4.1.6 (Heine-Borel). *Let Z be a subspace of $\mathbb{R}^n$. Z is compact if and only if Z is a bounded and closed subset.*

Proof. If Z is compact, then since $\mathbb{R}^n$ is Hausdorff, we know Z is closed according to proposition 4.1.3. Note that $Z \subset \bigcup_{r>0} B(0,r)$, so $Z \subset \bigcup_{r=r_1,\ldots,r_n} B(0,r)$, which means Z is bounded.

The reverse direction is direct since Z is a closed subspace of $[-N,N]^n$ for sufficiently large N. $\square$

4.2 Compact open topology

In this section, we consider the set of all continuous map from X to Y, dubbed $\mathcal{C}[X;Y]$, and assign a topology on it using compactness.

Definition 4.2.1 (Compact open topology). *Consider the set of all continuous map from X to Y: $\mathcal{C}[X;Y]$. Define a subbasis = $\{\mathcal{M}_{K,U} \mid K \subset X \text{ compact}, U \subset Y \text{ open}\}$, where*

$$\mathcal{M}_{K,U} = \{f \in \mathcal{C}[X;Y] \mid f(K) \subset U\}. \tag{19}$$

The topology generated by this subbasis is called a compact open topology.

Moreover, if (Y,d) is a metric topological space, then we have

Proposition 4.2.2. *(Y,d) metric space. A collection of sets*

$$B_K(f,\epsilon) = \{g \in \mathcal{C}[X;Y] \mid \sup_{x\in K} d(f(x),g(x)) < \epsilon\}. \tag{20}$$

with $K \subset X$ compact. Then $\{B_K(f,\epsilon)\}$ is a basis of compact open topology.

Example.

- The solution space for $\ddot{x} + x = 0$: $\mathcal{M} = \{x = a\cos t + b\sin t \mid (a,b) \in \mathbb{R}^2\}$ is homeomorphic to $\mathbb{R}^2$.

Proposition 4.2.3. *Y can be embedded into $\mathcal{C}[X;Y]$. Consider a bijective map $c : Y \to \mathcal{C}[X;Y]$ such that $y \mapsto f(X) = y$.*

Proposition 4.2.4. *$\mathcal{C}[X;Y]$ is Hausdorff if and only if Y is Hausdorff.*

Proposition 4.2.5. *$\mathcal{C}[X;\mathbb{R}]$ is path connected.*

Proof. $f : X \to \mathbb{R}$. Let $c_f : [0,1] \to \mathcal{C}[X;\mathbb{R}] : c_f(t) = t \cdot f$. $\square$

5 Homotopy

Homeomorphism is good for sure, but usually too restricted to be held. Here we discuss a weaker condition, i.e., Homotopy.

5.1 Homotopy of maps

Definition 5.1.1. *Let $f, g : X \to Y$ be two continuous maps. A homotopy between f and g is a continuous map $\sigma : X \times [0,1] \to Y$ such that $\forall x \in X$,*

$$\sigma(x,0) = f(x),\ \sigma(x,1) = g(x). \tag{21}$$

f and g is then homotopic, denoted $f \simeq g$.

Remark.

- σ defines a path in $\mathcal{C}[X;Y]$.
- Be cautious that σ must be continuous on $X \times [0,1]$. Only continuous for $\sigma(\cdot, t)$ is not enough.
- If f is homotopic to a constant function, it is called null-homotopic.

Proposition 5.1.2. *Homotopy $f \simeq g$ is an equivalence relation.*

Remark. We denote the quotient set $\mathcal{C}[X;Y]/\simeq$ as $[X;Y]$.

Proposition 5.1.3. *Let $f_1, f_2 : X \to Y$ and $g_1, g_2 : Y \to Z$. If $f_1 \simeq f_2$ and $g_1 \simeq g_2$, then*

$$g_1 \circ f_1 \simeq g_2 \circ f_2. \tag{22}$$

Proof Sketch. We prove by $g_1 \circ f_2 \simeq g_1 \circ f_1 \simeq g_2 \circ f_2$.

The first relation can be proved by $g_1 \circ \sigma_1$. The second can be shown by $\sigma_2'(x,t) = \sigma_2(f_2(x), t)$. □

5.2 Homotopy equivalence.

We consider the homotopic inverse. For $f : X \to Y$ we require its homotopic inverse to satisfy: $f \circ g \simeq Id_Y$ and $g \circ f \simeq Id_X$.

Definition 5.2.1 (Homotopy equivalence). *Topological spaces X and Y are called homotopy equivalent if there exist continuous maps $f : X \to Y$ and $g : Y \to X$ such that*

$$f \circ g \simeq Id_Y,\ g \circ f \simeq Id_X. \tag{23}$$

Examples.

- $\mathbb{R}^n \simeq \{x\}$.

Definition 5.2.2 (Contractible space). *X is a contractible space if $X \simeq \{x\}$. This is equivalent to the condition that Id_X is null-homotopic.*

Example.

- $\mathbb{R}\backslash\{0\} \simeq \{p_1\} \cup \{p_2\}$.
- $\mathbb{R}^2\backslash\{0\} \simeq S^1$.

To better describe the above observation, we define **defomation retraction**.

Definition 5.2.3 (Deformation retraction). *Let X be a topological space and A be a subspace of X. A deformation retraction is a homotopy*

$$r : X \times [0,1] \to X \tag{24}$$

such that $r(x,0) = x$ and $r(x,1) \in A$. Moreover, for any $a \in A$ and $t \in [0,1]$ we have $r(a,t) = a$.

5.3 Fundamental Group

Definition 5.3.1 (Loop). *Let X be a topological space and $\gamma : [0,1] \to X$ is a path on X. If $\gamma(0) = \gamma(1)$ we call it a loop.*

Definition 5.3.2 (Paths). *The inverse of a path γ is $\gamma^-(t) = \gamma(1-t)$. The constant path $I[x]$ is defined as $\gamma(t) = x$ for some $x \in X$.*

Definition 5.3.3 (Path homotopy). *Two paths with same initial and end points: $\gamma_1, \gamma_2 : [0,1] \to X$. $\gamma_1 \simeq_p \gamma_2$ if there exists a continuous map $\sigma : [0,1] \times [0,1] \to X$ with $\sigma(t,0) = \gamma_1(t)$ and $\sigma(t,1) = \gamma_2(t)$. Moreover, we need $\sigma(0,s) = \gamma_1(0) = \gamma_2(0)$ and $\sigma(1,s) = \gamma_1(1) = \gamma_2(1)$.*

Remark. Path homotopy is stricter than map homotopy as it fixes the initial and end points during deformation. To see their difference, consider $\mathbb{R}\backslash\{0\}$.

Definition 5.3.4 (Path multiplication). *Let α, β be paths on a topological space X. Suppose $\alpha(1) = \beta(0)$. We define*

$$(\alpha * \beta)(s) = \begin{cases} \alpha(2s), \ s \in [0, \frac{1}{2}], \\ \beta(2s-1), \ s \in [\frac{1}{2}, 1]. \end{cases} \tag{25}$$

Proposition 5.3.5. *For a path γ on X we have*

$$\gamma * \gamma^- \simeq_p I[\gamma(0)] \tag{26}$$

Note that $\alpha \simeq_p \alpha'$ and $\beta \simeq_p \beta'$ yields $\alpha * \beta \simeq_p \alpha' * \beta'$. We can define

$$[\alpha]_p * [\beta]_p = [\alpha * \beta]_p \tag{27}$$

Proposition 5.3.6. *Suppose α, β, γ are paths on X. Then*

$$[\alpha]_p * ([\beta]_p * [\gamma]_p) = ([\alpha]_p * [\beta]_p) * [\gamma]_p. \tag{28}$$

We now want to use define a group structure on this operation. However, not all paths on X can be multiplied as they might not have overlapped end points. To fix this issue, we consider loops with a fixed end points.

Definition 5.3.7 (Fundamental Group)**.** *Let X be a topological space. Let $x \in X$. We define*

$$\pi_1(X,x) = \{[\gamma]_p \mid \gamma \text{ loops in } X, \text{ i.e.}, \gamma(0) = \gamma(1) = x\}. \tag{29}$$

Then $(\pi_1(X,x), *)$ is a group.

Remark.

- On $\mathbb{R}^2$ all loops are path homotopic to the constant path. On $\mathbb{R}^2 \backslash \{0\}$, however, loops that wrap around 0 are not path homotopic to the constant path.
- A fundamental group is not necessarily Abelian.

Proposition 5.3.8. *$\pi_1((X,x),*)$ is a homotopic invariant.*

Examples.

- Let A be a band and B be a Mobius band. Then $\pi_1(A,*) \cong \pi_1(B,*) \cong \pi_1(S^1,*) \cong (\mathbb{Z},+)$.

Proposition 5.3.9. *If X is path connected, then $\pi_1((X,x),*) \cong \pi_1((X,y),*)$ for any $x, y \in X$.*

Remark. Let X be a topological space, then the followings are equivalent

- $\pi_1(X,x)$ is trivial
- $\forall \gamma_1, \gamma_2$ paths with $\gamma_1(0) = \gamma_2(0) = x$ and $\gamma_1(1) = \gamma_2(1)$, we have $\gamma_1 \simeq_p \gamma_2$.
- All loops at $x \simeq_p$ constant loop at x.

5.4 Covering Space

Definition 5.4.1 (Covering Space)**.** *Let (E,p) be a space over p with $p : E \to B$ continuous surjective. p is a covering map and (E,p) is a covering space over B, if $\forall b \in B$ there exists an open neighbor U of b such that*

$$p^{-1}(U) = \bigcup_{i \in I} U_i \tag{30}$$

where U_i are disjoint open subsets of E and $p|_{U_i} : U_i \to U$ is a homeomorphism.

Lemma 5.4.2 (lifting lemma)**.** *Let $p : E \to B$ be a covering map. Let $c : [0,1] \to B$ be a path on B. If $\tilde{b} \in E, p(\tilde{b}) = c(0) = b$, then there exists a unique path in E: $\tilde{c} : [0,1] \to E$, such that $\tilde{c}(0) = \tilde{b}$ and $p \circ \tilde{c} = c$.*

Proposition 5.4.3. *Let $p : E \to B$ be a covering map. Let $c_1, c_2 : [0,1] \to B$ be two homotopic paths on B. Suppose $c_1(0) = c_2(0) = b$. Then $\tilde{c}_1, \tilde{c}_2$ with $\tilde{c}_1(0) = \tilde{c}_2(0) = \tilde{b} \in p^{-1}(b)$ are also homotopic.*

5.5 Higher Homotopic Group

In discussion of path-connected components, we note that

$$\pi_0(X) = \{[x \to X] \mid x \in X\} \tag{31}$$

can be viewed as the set of all homotopic equivalent classes from a single point to X. Similarly,

$$\pi_1(X) = \{[[0,1] \to X] \mid x \in X\} \tag{32}$$

can be viewed as the set of all homotopic equivalent classes of loops in X. We want to generalize 0-dim point and 1-dim loops to n dimension in this section.

Definition 5.5.1 (n-cube). *$I_n = [0,1]^{\otimes n} = \{(t_1, \dots, t_n) \in \mathbb{R}^n \mid t_i \in [0,1]\}$. Then $\partial I_n = \{(t_1, \dots, t_n) \in I_n \mid \exists t_i = 0, 1\}$.*

Moreover, we define a n-loop based at $x \in X$ be a continuous map

$$\gamma : I_n \to X, \text{s.t., } \gamma|_{\partial I_n} = x. \tag{33}$$

Remark. This requirement is a generalization of the requirement $\gamma(0) = \gamma(1)$ in the 1-dimensional case, i.e., loops.

Definition 5.5.2 (Homotopy for n-loops). *Let γ, σ be two n-loops based at x. Let $\gamma(x) = \sigma(x)$ for any $x \in \partial I_n$. Then $\gamma \simeq \sigma$ homotopic as n-loops if there exists $H : I_n \times I \to X$ such that*

$$H(t;0) = \gamma(t), H(t;1) = \sigma(t), \text{ and } H(t;s) = x, \forall t \in \partial I_n, s \in I. \tag{34}$$

To construct group structures for n-loops, we define their composition as

$$\sigma_1 * \sigma_2 = \begin{cases} \sigma_1(2t_1, t_2, \dots, t_n), & 0 \le t_1 \le 1/2. \\ \sigma_2(2t_1 - 1, t_2, \dots, t_n), & \frac{1}{2} \le t_1 \le 1. \end{cases} \tag{35}$$

Remark.

- Surprisingly, $\sigma_1 * \sigma_2 \simeq \sigma_2 * \sigma_1$ for $n \ge 2$. In other words, $(\pi_n(X, x), *)$ is an Abelian group.
- Moreover, $[\sigma_1 * \sigma_2]_p$ well defined, in the sense that choosing the first index in eq. (35) is equivalent to choosing any other coordinate in $[n]$.

Examples.

- $\pi_n(S^n) \cong \mathbb{Z}, n \ge 1$.
- $\pi_5(S^2) \cong \mathbb{Z}_2, \pi_6(S^2) \cong \mathbb{Z}_{12}$.
- $\pi_1(\mathbb{R}P^2) \cong \mathbb{Z}_2, \pi_n(\mathbb{R}P^n) \cong \mathbb{Z}, n \ge 2$.
- (Bott periodicity of Lie groups) For $n \ge (k+1)/2$,

$$\pi_k(U(n)) \cong \pi_k(SU(n)) \cong \begin{cases} \{e\}, \ k \text{ even} \\ \mathbb{Z}, \ k \text{ odd}. \end{cases} \tag{36}$$

5.6 Application of Homotopy: Taxonomy of Defects

For a given material, describe its magnetic feature by its magnetic moment $\vec{m}(x) = \langle \sum_i m_i(x) \rangle$.

The free energy

$$\mathcal{F} = \mathcal{F}_{grad} + \mathcal{F}_{potential} + \mathcal{F}_{ext}. \tag{37}$$

where

$$\mathcal{F}_{grad} = \kappa |\nabla m|^2 \tag{38}$$

and

$$\mathcal{F}_{potential} = -\alpha(T)|m|^2 + \beta(T)|m|^4 + \frac{\alpha(T)^2}{4\beta(T)}. \tag{39}$$

Note that we should minimize $F = \int \mathcal{F}(x) \mathrm{d}x$. Suppose the gradient is zero, i.e., the magnetic moment is constant (homogeneous). If $\alpha(T) < 0, \beta(T) > 0$, then $|\vec{m}| = 0$. If $\alpha(T) > 0, \beta(T) > 0$, then $|\vec{m}| > 0$, i.e., Ferromagnetism.

We define defects as $\vec{m}(x), F < \infty$ where $\mathcal{F}$ is a local minimum. If $\vec{m}(x)$ is regular everywhere, we call it a regular defect. If $\vec{m}(x)$ has a singular point, we call it a singular defect.

For regular defects, we know when $|x| \to \infty$, $\mathcal{F} \to 0$ or otherwise the integration would not be finite. Thus,

$$\lim_{|x| \to \infty} |\vec{m}(x)| = m_0 \propto \sqrt{\frac{\alpha}{\beta}}. \tag{40}$$

The boundary condition for $\vec{m}(x) : \mathbb{R}^d \to \mathbb{R}^3$ is bounded to $\mathbb{S}^2$. The solution is locally stable guaranteed by homotopy on $\pi_{d-1}(\mathbb{S}^2)$.

$\pi_0(S^2) = \pi_1(S^2) = \{0\}$, $\pi_2(S^2) = \mathbb{Z}$.

6 Homology

6.0.1 Simplex

- 0-simplex: a point $\langle p_0 \rangle$.
- 1-simplex: an edge: $\langle p_0, p_1 \rangle$.
- 2-simplex: an triangle $\langle p_0, p_1, p_2 \rangle$, but p_0, p_1, p_2 do not lie on a same line.

Geometric independence: $r+1$ points are geometrically independent, if $p_0, \ldots, p_r \subseteq r - dim$ hyperplane, but $p_0, \ldots, p_r \not\subseteq (r-1) - dim$ hyperplane.

Definition 6.0.1 (r-simplex)**.** *$\{p_0, \ldots, p_r\}$ be $(r+1)$-geometrically independent. $p_0, \ldots, p_r \in \mathbb{R}^m$, $m \geq r$. A r-simplex $\sigma^r(\langle p_0, \ldots, p_r \rangle)$ is defined as*

$$\sigma^r = \{x \in \mathbb{R}^m \mid x = \sum_{i=1}^{r} c_i p_i, c_i \geq 0, \sum_{i=0}^{r} c_i = 1\} \tag{41}$$

Remark. $\{c_i\}$ are called the barycentric coordinates of x.

Definition 6.0.2 (q-face)**.** *Let $q \in \mathbb{Z}, 0 \leq q \leq r$. Select $q+1$ points from $\{p_0, \ldots, p_r\}$ and denote them by $\{p_{i_0}, \ldots, p_{i_q}\}$. Then the q-simplex $\langle p_{i_0}, \ldots, p_{i_q} \rangle$ is a q-face of $\langle p_0, \ldots, p_r \rangle$, denoted*

$$\langle p_{i_0}, \ldots, p_{i_q} \rangle \leq \langle p_0, \ldots, p_r \rangle \tag{42}$$

6.0.2 Simplicial Complex

Definition 6.0.3. *K is a set of simplexes. K is a simplicial complex if*

- $\sigma \in K, \sigma' \leq \sigma \Rightarrow \sigma' \in K$
- *For any $\sigma, \sigma' \in K$, either $\sigma \cap \sigma' = \varnothing$, or $\sigma \cap \sigma' \leq \sigma, \sigma \cap \sigma' \leq \sigma'$.*

Remark. For a simplicial complex K, the union of all elements in K is called a polyhedron, $|K| := \bigcup_{\sigma \in K} \sigma$.

Definition 6.0.4 (Triangulation)**.** *Let X be a topological space. If there exists a simplicial complex K, such that $f : |K| \to X$ is a homeomorphism, then X is triangulable, and (K, f) is a triangulation of X.*

6.1 Chain Group

Define orientation on $\langle p_1, \ldots, p_n \rangle$. Two simplexes have the same orientation if they differ by an even permutation.

Definition 6.1.1 (Chain group)**.** *K simplicial complex. The r-chain group on K, $C_r(K)$, is an Abelian group generated by the oriented r-simplexes of K. If $r > \dim K$, $C_r(K) = 0$. An element in $C_r(K)$ is called an r-chain.*

Remark.

$$C_r(K) = \{\sum_i z_i \sigma_i^r \mid z_i \in \mathbb{Z}, \sigma_i^r \in K\}. \tag{43}$$

- For $c = \sum_i c_i \sigma_i^r$, $d = \sum_i d_i \sigma_i^r$, $c + d = \sum_i (c_i + d_i) \sigma_i^r$.
- $C_r(K) \cong \mathbb{Z}^d$ for some d. d is the dimension of $C_r(K)$ and $d =$ the number of r-simplex in K.
- Consider $\dim K = 2$, $\dim C_2(K) - \dim C_1(K) + \dim_0(K)$ is the Euler characteristic for planar graphs, which is invariant under homotopy.

Next, we define cycle groups and boundary groups that are subgroups of chain groups. We start with defining what a boundary is.

Definition 6.1.2 (Boundary)**.** *Note that the boundary of a r-simplex is a union of $(r-1)$-simplexes it contains. We define a boundary operator $\partial_r : C_r(K) \to C_{r-1}(K)$:*

$$\partial_r(\sum_i z_i \sigma_i^r) = \sum_i z_i \partial_r(\sigma_i^r) \tag{44}$$

and for $\sigma_i^r = (p_0, p_1, \ldots, p_r)$,

$$\partial_r(\sigma_i^r) = \sum_{i=0}^{r} (-1)^i (p_0, \ldots, p_{i-1}, p_{i+1}, \ldots, p_r). \tag{45}$$

By definition we know σ_r is a group homomorphism.

Then the kernel $ker(\partial_r)$ is called the cycle group and the image $Im(\partial_r)$ is called the boundary group.

Definition 6.1.3 (Cycle group)**.** *An r-cycle group $Z_r(K)$ is defined as*

$$Z_r(K) = Ker(\partial_r) = \{c \in C_r(K) \mid \partial_r c = 0\}. \tag{46}$$

Definition 6.1.4 (Boundary group)**.** *An r-boundary group $B_r(K)$ is defined as*

$$B_r(K) = Im\partial_{r+1} = \{c \in C_r(K) \mid c = \partial_{r+1}d, d \in C_{r+1}(K)\}. \tag{47}$$

Proposition 6.1.5. *$\partial_r \circ \partial_{r+1} : C_{r+1}(K) \to C_{r-1}(K)$ is a zero map.*

Therefore, $B_r(K) \subset Z_r(K) \subset C_r(K)$.

6.2 Simplicial Homology

We define a **chain complex** $(C., d.)$ of Abelian groups $(C_n)_{n\in\mathbb{Z}}$ and homomorphisms of Abelian groups

$$d_n : C_{n+1} \to C_n \tag{48}$$

subject to $d_n \circ d_{n+1} = 0$ for all $n \in \mathbb{Z}$.

Definition 6.2.1 (Homology)**.** *An r-homology group of $(C., d)$ is defined as*

$$H_r(C., d) = \frac{Ker(d_r : C_r \to C_{r-1})}{Im(d_{r+1} : C_{r+1} \to C_r)} \tag{49}$$

Theorem 6.2.2. *Let X,Y be two triangulable homeomorphic topological space, and (K, f), (L, g) be triangulations of X and Y. Then*

$$H_r(K) \cong H_r(L) \tag{50}$$

Proposition 6.2.3. *Let K be a connected simplicial complex, then*

$$H_0(K) \cong \mathbb{Z} \tag{51}$$

6.3 Structure of Homology Group

Note that $H_r(K)$ is Abelian. The most general form of an Abelian group is

$$H_r(K) \cong \mathbb{Z}^n \times \mathbb{Z}_{k_1} \times \cdots \times \mathbb{Z}_{k_p} \tag{52}$$

The first part is called the free part and the second is called torsion part.

If we change the coefficient in chain groups from integer to real numbers, the torsion part disappears. So $H_r(K; \mathbb{R}) \cong \mathbb{R}^f$.

Definition 6.3.1 (Betti number)**.** *Let K be a simplicial complex. The r-th Betti number is*

$$b_r(K) = \dim H_r(K; \mathbb{R}) \tag{53}$$

which is also the rank of the free Abelian part of $H_r(K; \mathbb{Z})$.

Definition 6.3.2 (Euler characteristics)**.** *Let K be an n-dimensional simplicial complex, and I_r be the number of r-simplexes in K. The Euler characteristics χ is*

$$\chi(K) = \sum_{r=0}^{n} (-1)^r I_r. \tag{54}$$

Theorem 6.3.3 (Euler-Poincare)**.** *Let K be an n-dim simplicial complex,*

$$\chi(K) = \sum_{r=0}^{n} (-1)^r b_r(K). \tag{55}$$

6.4 Smith Normal Form

Let $A = (a_{ij}) \in M_{m\times n}(\mathbb{Z})$. A is in Smith normal form if

- $a_{ij} = 0$, $\forall i \neq j$
- $0 \leq r \leq \min(m, n)$, $a_{ii} \neq 0$ for $i \leq r$ and $a_{ii} = 0$ otherwise.
- $a_i = a_{ii}, 0 \leq i \leq r$, then $a_i | a_{i+1}$.

Proposition 6.4.1. *If $A \in M_{m\times n}(\mathbb{Z})$, then there exists $U \in M_{m\times m}(\mathbb{Z}), V \in M_{n\times n}(\mathbb{Z})$, $\det U = \pm 1, \det V = \pm 1$, and $D \in M_{m\times n}(\mathbb{Z})$ is in Smith normal form. We have*

$$A = UDV. \tag{56}$$

Proposition 6.4.2. *$C_{r-1}(K) \leftarrow C_r(K) \leftarrow C_{r+1}(K) \leftarrow \dots$. Let A be the matrix representation for ∂_{r+1} and B for ∂_r.*

Suppose $SNF(A) = \mathrm{diag}(a_1, \dots, a_r, \mathbf{0})$, then

$$H_r(K; \mathbb{Z}) \cong \mathrm{Ker} B / \mathrm{Im} A \cong \mathbb{Z}^{\dim C_r(K) - r - s} \times \mathbb{Z}_{a_1} \times \cdots \times \mathbb{Z}_{a_r} \tag{57}$$

where $r = \mathrm{rank} A, s = \mathrm{rank} B$.